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# Solving Delay Differential Equations through RBF Collocation

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**Abstract.** A general and easy-to-code numerical method based on radial basis functions (RBFs) collocation is proposed for the solution of delay differential equations (DDEs). It relies on the interpolation properties of infinitely smooth RBFs, which allow for a large accuracy over a scattered and relatively small discretization support. Hardy's multiquadric is chosen as RBF and combined with the Residual Subsampling Algorithm of Driscoll and Heryudono for support adaptivity. The performance of the method is very satisfactory, as demonstrated over a cross-section of benchmark DDEs, and by comparison with existing general-purpose and specialized numerical schemes for DDEs.

**AMS subject classifications**: 34-04, 65L99

**Key words**: Meshless method, delay differential equations, radial basis function, multiquadric, adaptive collocation.

## 1 Introduction

In this work, we present a general numerical approach for solving DDEs based on the RBF collocation method invented by Kansa [19] [20], also known as Kansa's method. Due to its many advantages (which include superior interpolation accuracy, spectral convergence, robustness with respect to the discretization support, and ease of coding), Kansa's method is becoming increasingly popular for the solution of ordinary and partial differential equations (ODEs and PDEs, respectively). Its performance in the solution of DDEs, however, has scarcely been explored, with the exception of a recent paper on the solution of neutral DDEs with multiquadrics [22].

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This paper is organized as follows. In Section 2, Kansa's method is adapted to a general formulation of (first order) DDEs. The basic algorithm is further improved by the inclusion of several heuristic observations concerning the tunable shape parameter which appears in the multiquadric RBF, and by the residual subsampling algorithm (RSA) by Driscoll and Heryuodono [8]. The RSA is at the core of the high accuracy attained by the multiquadrics interpolant. Section 2 is closed by some remarks concerning the solution of nonlinear problems with Kansa's method. Section 3 tests the proposed method against a cross-section of benchmark problems taken from the literature. As we shall see, not only does Kansa's method attain excellent results in well-understood (first order) DDEs, but also in the less explored neutral and higher-order DDEs — which may offer an additional tool for looking into this kind of problems. Finally, Section 4 concludes the paper.

## 2 Solving linear DDEs through Kansa's method

Consider the following linear DDE

$$y'(x) - p(x)y(x) - q(x)y[x-\tau(x)] = s(x), \qquad \text{if } x \in (a,b], \tag{2.1}$$

$$y(x) = h(x), \qquad \text{if } x \le a. \tag{2.2}$$

It will be convenient to split (2.2) into a DDE and an ODE

$$y'(x) - p(x)y(x) - q(x)y[x-\tau(x)] = s(x), \qquad \text{if } x-\tau(x) > a, \tag{2.3}$$

$$y'(x) - p(x)y(x) = q(x)h[x-\tau(x)] + s(x), \qquad \text{if } x-\tau(x) < a, \tag{2.4}$$

$$y(a) = h(a). \tag{2.5}$$

Discretize $[a,b]$ into a set $N$ scattered nodes $\xi = \{x_j, j = 1...N\}$ (with $x_1 = a$ and $x_N = b$), and consider as well the outside point $x_0 = a - \lambda$, $\lambda > 0$. We seek an approximate solution to (2.3)-(2.5) in the form of an expansion of $N+1$ RBFs $\phi_j(r)$:

$$y(x) = \sum_{j=0}^{j=N} \alpha_j \phi(\|x - x_j\|). \tag{2.6}$$

The addition of an RBF at $x_0$ allows to enforce *both* the initial condition *and* the DDE at $x = a$, thus contributing to the accuracy (this is the PDECB strategy discussed in [12]). Once the coefficients $\alpha_j$ are available, the approximate RBF solution can be reconstructed anywhere in $[a,b]$. In order to solve for the coefficients, (2.3)-(2.5) are enforced over (2.6) on a set of collocation $N$ nodes, usually $\xi$. Notice that no equation is collocated on $x_0$, but two of them are on $x_1 = a$. For $i = 1, \ldots, N$, this leads to the linear system of dimension $N+1$,

$$\sum_{j=0}^{j=N} \Big\{ \phi_j'(r_{ij}) - p(x_i)\phi_j(r_{ij}) - q(x_i)\phi_j(\|x_i - \tau(x_i) - x_j\|) \Big\} = s(x_i),$$
$$\text{if } x_i - \tau(x_i) > a, \tag{2.7}$$