Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **2**, No. 2, pp. 211-221

DOI: 10.4208/aamm.09-m0967
April 2010

Valuation of American Call Option Considering Uncertain Volatility

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Received 20 July 2009; Accepted (in revised version) 30 September 2009
Available online 5 March 2010

Abstract. The parabolic variational inequality for simulating the valuation of American option is used to analyze a continuous dependence of the solution with respect to the uncertain volatility parameter. Three kinds of the continuity are proved, enabling us to employ the maximum range method for the uncertain parameter, under the condition that the criterion-functional has the corresponding property.

AMS subject classifications: 91B28, 49J40, 49N30.
Key words: American options, parabolic variational inequality, uncertain parameter.

1 Introduction

The problem of pricing American options is important both in theory and in practice. It has been shown by the Nobel Prize laureates Merton [7] and Black and Scholes [4] that the valuation of American call option can be simulated by a free boundary problem for a degenerate parabolic equation. A weak solution of the problem has been defined by Badea and Wang [2]. They proved the existence and uniqueness of the weak solution and some regularity results by a detailed analysis based on the use of maximum principles.

Efficient numerical methods for the solution of the problem using finite elements in "space" and backward differences in "time" have been proposed by Allegretto et al. [1] and Lin et al. [8]. These authors started from an equivalent variational inequality, which can be derived by a suitable change of all variables and which avoids the degeneracy.

The aim of the present paper is to complete the results of Badea and Wang [2, 3] by an analysis of a continuous dependence of the weak solution with respect to the volatility. The latter parameter appears to be the only parameter, which is not observable directly in the market. On the basis of a variational inequality for the weak

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solution [3], we prove three kinds of the continuous dependence, provided the volatility belongs to a given compact interval of positive numbers. Then one can employ the maximum range method [5,6], if a criterion-functional is prescribed, which has properties, corresponding to the three kinds of the continuity, mentioned above.

2 Formulation by a parabolic variational inequality

The original mathematical model of American call option proposed by Merton [7] is represented by a free boundary problem for a parabolic equation

$$w_t - \frac{1}{2}\sigma^2 x^2 w_{xx} - (r-d)xw_x + rw = 0, \qquad \text{in } D, \tag{2.1}$$

with the initial condition

$$w(x,0) = 0,$$

and the boundary conditions

$$w(0,t) = 0, \quad w\big(s(t),t\big) = (x-Z)^+, \quad w_x\big(s(t),t\big) = 1,$$

for all $t \in (0,T]$. Here $w(x,t)$ denotes the value of the American call option, T is the maturity date (the time at which the American call option expires), r is the interest rate, d the dividend rate, σ the volatility, Z the exercise price, x denotes the stock price, $t \equiv T - t_r$, where t_r is the real time,

$$D = \Big\{(x,t):\ 0 < x < s(t), \quad t \in (0,T]\Big\}.$$

The free boundary $x = s(t)$ denotes the optimal exercise curve,

$$w_x \equiv \frac{\partial w}{\partial x}, \quad w_t \equiv \frac{\partial w}{\partial t}, \quad \text{and} \quad (u)^+ = \max\{u,0\}.$$

We assume that r, d, Z, σ are positive real constants.

If we define a new function $u(x,t)$ by

$$u = w - (x-Z)^+,$$

we can extend the function u outside the domain D by zero. In this way a weak solution has been defined by Badea and Wang in [2], where an upper bound

$$S_0(\sigma) = \frac{Z\lambda(\sigma)}{\big(\lambda(\sigma)-1\big)},$$

with

$$\lambda(\sigma) = \sigma^{-2}\Big\{\frac{\sigma^2}{2} - r + d + \Big[\Big(\frac{\sigma^2}{2} - r + d\Big)^2 + 2r\sigma^2\Big]^{\frac{1}{2}}\Big\},$$