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Generating Function Methods for Coefficient-Varying Generalized Hamiltonian Systems

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Abstract. The generating function methods have been applied successfully to generalized Hamiltonian systems with constant or invertible Poisson-structure matrices. In this paper, we extend these results and present the generating function methods preserving the Poisson structures for generalized Hamiltonian systems with general variable Poisson-structure matrices. In particular, some obtained Poisson schemes are applied efficiently to some dynamical systems which can be written into generalized Hamiltonian systems (such as generalized Lotka-Volterra systems, Robbins equations and so on).

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1 Introduction

We consider the generalized Hamiltonian systems (cf. [3,5,9])

$$y'(t)=B(y)\nabla H(y), \qquad y=(y_1,y_2,\cdots,y_n)^T\in\mathcal{M}, \tag{1.1}$$

where $\mathcal{M}$ is a differential manifold in $\mathbb{R}^n$, ∇ is the gradient operator, $H\in\mathbf{C}^\infty(\mathcal{M})$ is a Hamiltonian function, $B(y)=(b_{ij}(y))_{i,j=1}^n$ is a skew-symmetric Poisson-structure matrix and satisfies the Jacobi identity

$$b_{ij}(y)\frac{\partial b_{lk}(y)}{\partial y_i}+b_{ik}\frac{\partial b_{jl}(y)}{\partial y_i}+b_{il}(y)\frac{\partial b_{kj}(y)}{\partial y_i}=0, \qquad i,j,k,l=1,2,\cdots,n.$$

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The corresponding Poisson bracket (cf. [16,17]) is defined as

$$\{F,H\}(y)=(\nabla F(y))^T B(y)\nabla H(y), \quad \forall F,H\in \mathbf{C}^\infty(\mathcal{M}).$$

Definition 1.1 (cf. [5], Chapter 12)**.** A map $y\to\hat{y}=g(y)$: $\mathcal{M}\to\mathcal{M}$ is called a Poisson map, if it is a (local) diffeomorphism and preserves the Poisson bracket, i.e.,

$$\{F\circ g,H\circ g\}=\{F,H\}\circ g, \quad \forall F,H\in \mathbf{C}^\infty(\mathcal{M}).$$

An n-order square matrix $M(y,\hat{y})$ is called a Poisson matrix if

$$M(y,\hat{y})B(y)M(y,\hat{y})^T=B(\hat{y}).$$

A function $C(y)\in\mathbf{C}^\infty(\mathcal{M})$ is called a Casimir function if

$$\{C,F\}(y)=0, \quad \forall F\in \mathbf{C}^\infty(\mathcal{M}).$$

It is easy to prove that $g(y)$ is a Poisson map if and only if

$$g_y(y)B(y)(g_y(y))^T=B(\hat{y}).$$

For a numerical algorithm applied to the systems (1.1), we hope that it can preserve more structure characterizations of the original systems. If the discrete flow obtained by an algorithm for the systems (1.1) is a Poisson map, then we say this algorithm is a Poisson-structure-preserving algorithm, referred to a Poisson scheme. And the Poisson scheme is also an extension of the symplectic algorithm (cf. [2,5,14,18]).

Generating function methods (cf. [5,8,11,15]) are also important approaches to construct the symplectic scheme for canonical Hamiltonian systems and the Poisson schemes for generalized Hamiltonian systems. So far, some generating function methods preserving the Poisson structures for linear generalized Hamiltonian systems (i.e., Lie-Poisson systems) and the generalized Hamiltonian systems with constant or invertible Poisson-structure matrices have been presented respectively (cf. [6,7,19]). Moreover, in this paper, we extend these results and present the generating function methods for generalized Hamiltonian systems with general variable Poisson-structure matrices which can be singular. In particular, the obtained Poisson schemes are applied efficiently to some dynamical systems which can be written into the forms of generalized Hamiltonian systems (such as generalized Lotka-Volterra systems, Robbins equations and so on).

In Section 2, we extend the Hamilton-Jacobian theorem to the coefficient-varying generalized Hamiltonian systems (1.1), and get their generating functions. In Section 3, based on the obtained results, we construct some Poisson schemes for the systems (1.1). In Section 4, we use these obtained schemes to solve several specific systems and give the corresponding numerical results.