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Lower Bounds for Eigenvalues of the Stokes Operator

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Abstract. In this paper, we propose a condition that can guarantee the lower bound property of the discrete eigenvalue produced by the finite element method for the Stokes operator. We check and prove this condition for four nonconforming methods and one conforming method. Hence they produce eigenvalues which are smaller than their exact counterparts.

AMS subject classifications: 65N30, 65N15, 35J25

Key words: Lower bound, eigenvalue, Stokes operator.

1 Introduction

We are interested in the lower bound property of the eigenvalue by the (conforming and nonconforming) finite element method for the Stokes operator. We propose a condition that can guarantee theoretically the lower bound property of the discrete eigenvalue for both conforming and nonconforming methods. We check and prove this condition for the nonconforming rotated Q_1 element [20], the enriched nonconforming rotated Q_1 element [16], the Crouzeix-Raviart element [9] and the enriched Crouzeix-Raviart element [11] and the conforming P_2-P_0 element.

The lower bound property of the eigenvalue by nonconforming methods of the Stokes eigenvalue problem was first analyzed in [17]. We here give a new error estimate for eigenvalues and eigenfunctions and slightly different analysis for the lower bound property. For the conforming element, we present the first analysis of the lower bound property of the discrete eigenvalue.

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The analysis herein will use some identity for the error of the eigenvalue. Such type of an identity was first analyzed in a remarkable paper Armentano and Duran [1] for the nonconforming linear element of the Laplace operator. The idea was independently extended to the Wilson element in Zhang et al. [25] and to the enriched nonconforming rotated Q_1 element in Li [14]. In those papers, canonical interpolation operators of these nonconforming elements were performed. For the nonconforming linear element of the Laplace operator, there is some special projection property for the canonical interpolation operator, namely, the interpolation is identical to the Galerkin projection. However, for the general case, one has not such a special projection property for the canonical interpolation operator, see Zhang, Yang et al. [25] and Li [14]. Therefore, that term will not be zero any more. From arguments in [1, 14, 25], it is straightforward to see that a similar identity holds for any function (not necessary the canonical interpolation) in the nonconforming finite element space, we refer interested readers to, Yang et al. [23] and Hu et al. [11] for more details. This idea was extended to the Stokes operator in Lin et al. [17], which will be used in the present paper.

We end this section by introducing necessary notation. We use the standard gradient operator:

$$\nabla r := \Big(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}\Big).$$

Given any $2D$ vector function $\psi = (\psi_1, \psi_2)$, its divergence reads $\operatorname{div}\psi := \partial\psi_1/\partial x + \partial\psi_2/\partial y$. The spaces $H_0^1(\Omega)$ and $L_0^2(\Omega)$ are defined as usual,

$$\begin{aligned} &H_0^1(\Omega) := \big\{v \in H^1(\Omega),\ v = 0 \text{ on } \partial\Omega\big\},\\ &L_0^2(\Omega) := \Big\{q \in L^2(\Omega),\ \int_\Omega dx = 0\Big\}. \end{aligned}$$

Suppose that $\overline{\Omega}$ is covered exactly by a shape-regular triangulation $\mathcal{T}_h$ consisting of triangles in $2D$, see [8]. Let $\mathcal{E}_h$ be the set of all edges in $\mathcal{T}_h$, $\mathcal{E}_h(\Omega)$ the set of interior edges and $\mathcal{E}(K)$ the set of edges of any given element K in $\mathcal{T}_h$; $h_K = |K|^{1/2}$, the size of the element $K \in \mathcal{T}_h$, where $|K|$ is the area of element K. ω_K is the union of elements $K' \in \mathcal{T}_h$ that share an edge with K and ω_E is the union of elements that share a common edge E. Given any edge $E \in \mathcal{E}(\Omega)$ with the length h_E we assign one fixed unit normal $\nu_E := (\nu_1, \nu_2)$ and tangential vector $\tau_E := (-\nu_2, \nu_1)$. For E on the boundary we choose $\nu_E = \nu$ the unit outward normal to Ω. Once ν_E and τ_E have been fixed on E, in relation to ν_E one defines the elements $K_- \in \mathcal{T}_h$ and $K_+ \in \mathcal{T}_h$, with $E = K_+ \cap K_-$. Given $E \in \mathcal{E}(\Omega)$ and some $\mathbb{R}^d$-valued function v defined in Ω, with $d = 1,2$, we denote by $[v] := (v|_{K_+})|_e - (v|_{K_-})|_E$ the jump of v across E where $v|_K$ denote the restriction of v on K.

The paper is organized as follows. In the following section, we shall present the Stokes eigenvalue problem and its finite element methods in an abstract setting. In Section 3, based on two conditions on the discrete spaces, we analyze error estimates for both discrete eigenvalues and eigenfunctions. In Section 4, under one more condition,