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Two-Scale Picard Stabilized Finite Volume Method for the Incompressible Flow

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Abstract. In this paper, we consider a two-scale stabilized finite volume method for the two-dimensional stationary incompressible flow approximated by the lowest equal-order element pair P_1-P_1 which do not satisfy the inf-sup condition. The two-scale method consist of solving a small non-linear system on the coarse mesh and then solving a linear Stokes equations on the fine mesh. Convergence of the optimal order in the H^1-norm for velocity and the L^2-norm for pressure are obtained. The error analysis shows there is the same convergence rate between the two-scale stabilized finite volume solution and the usual stabilized finite volume solution on a fine mesh with relation $h=\mathcal{O}(H^2)$. Numerical experiments completely confirm theoretic results. Therefore, this method presented in this paper is of practical importance in scientific computation.

AMS subject classifications: 35Q10, 65N30, 76D05

Key words: Incompressible flow, stabilized finite volume method, inf-sup condition, local Gauss integral, two-scale method.

1 Introduction

Finite volume method (FVM) is an important numerical tool for solving partial differential equations. It has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer, and petroleum engineering. The FVM is intuitive in that it is directly based on local conservation of mass, momentum, or energy over volumes (control volumes or co-volumes). It lies somewhere between the finite element methods (FEM) and the finite difference methods (FDM) and has the flexibility similar to that of the

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FEM for handling complicated geometries. Implementation is comparable to that of the FDM. The FVM is also referred to as the control volume method, the covolum method, or the first order generalized difference method [4, 6, 8–10, 37]. Its theoretical analysis is much more complex than that of the FEM.

In the paper [18], the authors consider the finite volume algorithm for solving the S-tokes problems and shows that it has optimal efficiency for the velocity in L^2-norm in the sense that if $f \in [H^1(\Omega)]^d$, $d=2,3$. A superconvergence result is established for the stationary Navier-Stokes equations by a stabilized finite volume method and L^2-projection on a coarse mesh [16, 30, 31]. Then a new stabilized FVM is studied and developed by Li et al. in [30] for the stationary incompressible flow. This method is based on a local Gauss integration technique and uses the lowest equal order finite element pair P_1-P_1 that do not satisfy the inf-sup (LBB) stability conditions [13, 22, 24, 25]. Stability and convergence of the optimal order in the L^2-norm and H^1-norm for velocity and the L^2-norm for pressure are obtained. A new duality for the incompressible flow is introduced to establish the convergence of the optimal order in the L^2-norm for velocity [19, 30].

Two-scale schemes have been applied to a variety of the steady semilinear equation-s by Xu [35, 36], the steady non-linear saddle point problems with the non-linear constraints by Niemisto in his thesis [33], and the steady incompressible flow by Layton, Li and Hou [17,20,23,26–29], and later on by Girault and Lions with particular emphasis on the three-dimensional problem on domains with corners [11].At the same time, Chen and Liu [7] have also studied this method for semilinear parabolic problems. However, more study is required for the stationary incompressible flow of finite volume approximation. Moreover, the theoretic analysis of two-scale stabilized FVM is more difficult than that of FEM.

In this article, we combine stabilized FVM based on P_1-P_1 element with two-scale strategy to obtain a two-scale stabilized FVM for the two-dimensional incompressible flow.The main procedure is stated as follows:

Step 1 Solve a small non-linear system on the coarse scale.

Step 2 Solve a linear system on the fine scale.

The convergence of the optimal order in the H^1-norm for velocity and L^2-norm for pressure are obtained. We choose the two-scale spaces as two conforming finite element spaces V_H, Q_H and V_h, Q_h on one coarse grid with mesh size H and one fine grid with mesh size $h \ll H$. The two-scale method consist of solving a small incompressible flow problem on the coarse mesh and then solving a linear Stokes problem on the fine mesh. Then, we prove that the two-scale stabilized finite volume solution (u^h, p^h) has the following error estimate:

$$\|\nabla(u-u^h)\|_0+\|p-p^h\|_0 \le C(h+H^2).$$

In solving approximate solution of the stationary incompressible flow on a fine mesh sizes satisfying $h=\mathcal{O}(H^2)$, the error analysis shows that the two-scale stabilized FVM