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Nonpolynomial Jacobi Spectral-Collocation Method for Weakly Singular Fredholm Integral Equations of the Second Kind

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Abstract. In this paper a nonpolynomial Jacobi spectral-collocation (NJSC) method for the second kind Fredholm integral equations (FIEs) with weakly singular kernel $|s-t|^{-\gamma}$ is proposed. By dividing the integral interval symmetrically into two parts and applying the NJSC method symmetrically to the two weakly singular FIEs respectively, the mild singularities of the interval endpoints can be captured and the exponential convergence can be obtained. A detailed L^∞ convergence analysis of the numerical solution is derived. The NJSC method is then extended to two dimensional case and similar exponential convergence results are obtained for low regular solutions. Numerical examples are presented to demonstrate the efficiency of the proposed method.

AMS subject classifications: 65L70, 45B05

Key words: Nonpolynomial Jacobi spectral-collocation method, Fredholm integral equations, weakly singular, exponential convergence.

1 Introduction

Fredholm integral equations of the second kind often arise in practical applications such as astrophysics, mathematical problems of radiative equilibrium, electrical engineering and radiative heat transfer problems [1,35–37]. In this paper, we consider weakly singular Fredholm integral equations (FIEs) of the second kind

$$u(t)=g(t)+\int_I k(t,s)u(s)ds, \quad t\in I=[0,1], \tag{1.1}$$

where the function $g(t)\in C(I)$, $u(t)$ is the solution to be determined, the kernel function $k(t,s)$ is weakly singular.

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The weakly singular FIE of the second kind usually has a solution with mild singularities at the boundary of solving domain. This integral can be discretized using generalized Gaussian quadratures [48] which take into account the nature of the integrand as well as the precise geometry of the interaction list. However, this approach is far from optimal in terms of the number of quadrature nodes needed for achieving a pre-selected desirable precision in higher dimensions [29]. We would like to remark here the recursively compressed inverse preconditioning (RCIP) method, as an efficient and accurate treatment at corner singularities based on the Nyström method [14], albeit with a slightly lower accuracy. In order to deal with the mild singularities, another frequently used technology is the graded mesh and there are considerable excellent relevant research works such as [13, 20, 36, 43]. However, the use of graded mesh has a practical limitation which serious round off errors occur when the initial stepsize becomes very small. To avoid this problem, many other methods have been proposed. For example, Tang [41] used spline collocation methods to solve Volterra integro-differential equations with weakly singular kernels by suitable graded mesh and obtained the optimal convergence. Cao and Xu [7] developed a singularity preserving Galerkin method in which some non-polynomial functions reflecting the singularity of the exact solution and quasi-uniform partitions are used to avoid round-off errors. Furthermore, Cao et al. [6] presented numerical solutions of weakly singular FIEs of the second kind by hybrid collocation method that preserves the singularity of the exact solution and at the same time provides the optimal order of convergence. Wang [44] further developed hybrid multistep collocation method for the weakly singular FIEs, which converges faster with lower degrees of freedom and more efficiently captures the weakly singular properties by nonpolynomial interpolation at the first subinterval. Smoothing transformation that transform the current weakly singular solution to a smoother one is also a popular strategy, see, for example [25, 30–32]. As an efficient method, Galerkin method has been applied to solve Volterra integral equations with weakly singular kernels numerically. Yi and Guo [49] presented an h-p version of the continuous Petrov-Galerkin (CPG) finite element method for linear Volterra integro-differential equations with smooth and nonsmooth kernels. Wang et al. [46] developed and analyzed an hp-version of the discontinuous Galerkin time-stepping method for linear Volterra integral equations with weakly singular kernels. Yi et al. [50] proposed a very simple but efficient postprocessing technique for improving the global accuracy of the discontinuous Galerkin (DG) time stepping method for solving nonlinear Volterra integro-differential equations by adding a higher order generalized Jacobi polynomial of degree $k+1$ with parameters $(-1,0)$ to the DG approximation of degree k. In addition, postprocessing method, including interpolation postprocessing and iteration postprocessing, is a common accelerated convergence technique. Huang and Zhang [19] discussed the superconvergence of the interpolated collocation solutions for Hammerstein equations. Then Huang and Wang [18] discussed the superconvergence of the interpolated postprocessing method for weakly singular Volterra integral equations (VIEs) of the second kind based on collocation method and hybrid collocation method. Graham [12] obtained the numerical solutions by the collocation and iterated collocation methods of