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A New Well-Balanced Finite Volume CWENO Scheme for Shallow Water Equations over Bottom Topography

Wei Guo[1], Ziming Chen[1], Shouguo Qian[1,*], Gang Li[1,*] and Qiang Niu[2]

[1] *School of Mathematics and Statistics, Qingdao University, Qingdao, Shandong 266071, China*

[2] *Department of Applied Mathematics, Xi'an Jiaotong–Liverpool University, Suzhou, Jiangsu 215123, China*

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Abstract. In this article, we develop a new well-balanced finite volume central weighted essentially non-oscillatory (CWENO) scheme for one- and two-dimensional shallow water equations over uneven bottom. The well-balanced property is of paramount importance in practical applications, where many studied phenomena can be regarded as small perturbations to the steady state. To achieve the well-balanced property, we construct numerical fluxes by means of a decomposition algorithm based on a novel equilibrium preserving reconstruction procedure and we avoid applying the traditional hydrostatic reconstruction technique accordingly. This decomposition algorithm also helps us realize a simple source term discretization. Both rigorous theoretical analysis and extensive numerical examples all verify that the proposed scheme maintains the well-balanced property exactly. Furthermore, extensive numerical results strongly suggest that the resulting scheme can accurately capture small perturbations to the steady state and keep the genuine high-order accuracy for smooth solutions at the same time.

AMS subject classifications: 74S10

Key words: Shallow water equations, source term, CWENO scheme, decomposition algorithm, well-balanced property.

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1 Introduction

In this article, we concern with developing a high-order finite volume CWENO scheme for shallow water equations (SWEs), which in one-dimensional space enjoy the following

*Corresponding author.
Emails: 1161362621@qq.com (W. Guo), zimingchen123@163.com (Z. Chen), sgqian1976@163.com (S. Qian), gangli1978@qdu.edu.cn (G. Li), qiang.niu@xjtlu.edu.cn (Q. Niu)

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form

$$h_t+(hu)_x=0, \tag{1.1a}$$

$$(hu)_t+\left(hu^2+\frac{1}{2}gh^2\right)_x=-ghb_x. \tag{1.1b}$$

Here, $h(x,t)$, $u(x,t)$ and $b(x)$ represent the water depth, the depth-averaged velocity, and the bottom, respectively. The constant $g=9.812$ denotes the gravitational constant. High order schemes solving SWEs enjoy the key role in fields of the hydraulic science and the coastal engineering [1–3].

The geometric source term on the right hand side of (1.1) is due to the uneven bottom topography. For conciseness, the system (1.1) can be rewritten as a compact vector form

$$U_t+F(U)_x=S(U,b), \tag{1.2}$$

with

$$U=(h,hu)^\top, \quad F(U)=\left(hu,hu^2+\frac{1}{2}gh^2\right)^\top \quad \text{and} \quad S(U,b)=(0,-ghb_x)^\top$$

being the vector of the conservative variable, the physical flux, and the source term, respectively. The system (1.1) keeps a subtle equilibrium between the flux gradient and the source term. From the mathematical point of view, the system (1.2) holds non-trivial steady state solutions that satisfy an ordinary differential equation (ODE)

$$F(U)_x=S(U,b).$$

Herein, we are interested in the still water steady state solutions as follows

$$u=0, \quad h+b=\text{Constant}. \tag{1.3}$$

In general, the traditional schemes coupled with standard numerical fluxes as well as direct discretizations of the source term fail to maintain the above delicate equilibrium, and lead to non-physical oscillations especially near discontinuities, which will not disappear even on a very refined mesh.

Well-balanced schemes [4, 5] can preserve the steady state up to the machine accuracy at the discrete level and resolve small perturbations of the steady state even on a relatively coarse mesh [6], then increase the computational efficiency correspondingly. Following the original works [4, 5], many researchers have made extensive attempts in the development of well-balanced schemes. Representative researches mainly include: kinetic scheme [7], gas-kinetic scheme [8], central-upwind scheme [9], weighted essentially non-oscillatory (WENO) schemes [10–19], Hermite WENO scheme [20], central schemes [21, 22], Runge-Kutta discontinuous Galerkin (RKDG) methods [23–25], ADER (Arbitrary DERivatives in time and space) schemes [26,27], spectral element method [28], Godunov-type method [29], element-free Galerkin method [30], ADER discontinuous