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Sinc-Multistep Schemes for Forward Backward Stochastic Differential Equations

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Abstract. In this work, by combining the multistep discretization in time and the Sinc quadrature rule for approximating the conditional mathematical expectations, we will propose new fully discrete multistep schemes called "Sinc-multistep schemes" for forward backward stochastic differential equations (FBSDEs). The schemes avoid spatial interpolations and admit high order of convergence. The stability and the K-th order error estimates in time for the K-step Sinc multistep schemes are theoretically proved ($1\le K\le 6$). This seems to be the first time for analyzing fully time-space discrete multistep schemes for FBSDEs. Numerical examples are also presented to demonstrate the effectiveness, stability, and high order of convergence of the proposed schemes.

AMS subject classifications: 65C30, 60H10, 60H35

Key words: Forward backward stochastic differential equations, multistep schemes, Sinc quadrature rule, error estimates.

1 Introduction

This paper is concerned with the numerical solution of the following FBSDEs defined on a filtered complete probability space $(\Omega,\mathcal{F},\mathbb{F},P)$:

$$\begin{cases} X_t = X_0 + \int_0^t b(s,X_s)dt + \int_0^t \sigma(s,X_s)dW_s, & \text{(SDE)} \\ Y_t = \varphi(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, & \text{(BSDE)} \end{cases} \tag{1.1}$$

for $t\in[0,T]$, where the natural filtration $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ is generated by the standard d-dimensional Brownian motion $W=(W_t)_{0\le t\le T}$. $X_0\in\mathcal{F}_0$ is the initial condition of forward SDE, $b:\ \Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ are referred to the drift and

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diffusion coefficients, respectively; $\varphi: \mathbb{R}^d \to \mathbb{R}^m$ is the terminal function of X_T for BSDE, and $f: \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ is the generator of BSDE. $(X_t, Y_t, Z_t): [0,T] \times \Omega \to \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ is the unknown. A triple (X_t, Y_t, Z_t) is called an L^2-adapted solution of the FBSDEs (1.1) if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies (1.1).

Lots of efforts have been devoted to the study of the FBSDEs due to their applications in various fields, such as mathematical finance [9], stochastic optimal control [18], the theory of partial differential equations (PDEs) [17] and so on. As the FBSDEs seldom admit explicitly closed-form solutions, numerical methods have played an important role in applications. Up to now, there exists a very extensive body of literature in numerical methods for FBSDEs. For the temporal discretizations, popular strategies include the Euler-type methods [4, 8, 26], the generalized θ-schemes [27] and multistep schemes [3, 28], to just name a few. For the spatial discretizations, the accuracy and the efficiency of approximations of conditional mathematical expectations are essential in numerically solving FBSDEs. A partial list can be roughly classified into grid-type methods [13,30,31], or regression methods [12]. The main difficulty in the mentioned gird-based methods is to design the corresponding high-dimensional quadrature rules and the high-accurate spatial interpolations.

Recently, a novel kind of linear multistep methods for FBSDEs, whose origin can be traced back to [28], has attracted increasingly attentions, since it was shown for FBSDEs models that their numerical solutions of the BSDE maintain the high-order accuracy even if the Euler method is used to solve the forward SDE. This idea has been successfully extended to a broader class of FBSDEs and their related problems: FBSDEs with jumps [6], second order FBSDEs [30], stochastic optimal control problems [7] and nonlinear PDEs [10]. Under certain hypothesis with the exact solution of the forward SDE given, the stability for linear multistep schemes for BSDEs was proved in [3], and the error estimates of the multistep schemes for FBSDEs with f independent of Z were given in [24]. A modified explicit linear multistep scheme for BSDEs was proposed and analyzed in [22]. To the best of our knowledge, the error analysis for fully discrete multistep schemes for FBSDEs is still lacking.

We in this work mainly focus on proposing new highly accurate fully discrete multistep schemes and analyzing them. The main contributions of this paper are as follows.

- By approximating the conditional expectations via the Sinc quadrature rule, we propose new high-order fully discrete multistep schemes called "Sinc-multistep schemes" for decoupled FBSDEs. The most important feature of the schemes is that no spatial interpolations are required by using integral variable transformations and by properly choosing space step parameters in the Sinc quadrature rule.

- By applying numerical algebra theory and the properties of the approximate operator, we theoretically obtain the optimal Kth-order of convergence for the K-step fully discrete multistep scheme for the FBSDEs (1.1) with the drift term $b = 0$ and the diffusion term σ a nonsingular matrix.