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Unconditional Superconvergence Analysis of Energy Conserving Finite Element Methods for the Nonlinear Coupled Klein-Gordon Equations

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Abstract. In this paper, we consider the energy conserving numerical scheme for coupled nonlinear Klein-Gordon equations. We propose energy conserving finite element method and get the unconditional superconvergence result $\mathcal{O}(h^2+\Delta t^2)$ by using the error splitting technique and postprocessing interpolation. Numerical experiments are carried out to support our theoretical results.

AMS subject classifications: 65N06, 65B99

Key words: Energy conserving, the nonlinear coupled Klein-Gordon equations, unconditional superconvergence result, postprocessing interpolation, finite element method.

1 Introduction

Nonlinear equation theory, especially nonlinear Klein-Gordon equations, is an important subject, which is widely used in mathematical physics equations and chemical kinetics. In recent years, the nonlinear Klein-Gordon equations have been widely used in many scientific fields such as fluid dynamics, plasma physics, nonlinear optics, and other research areas (see [1–7] and the related references therein).

In theoretical and numerical approach, lots of works about the nonlinear Klein-Gordon equations have been studied. In [7], R. K. Dodd et al. proved the existence, uniqueness, scatter and asymptotic behavior for these type wave equations. The system of coupled Klein-Gordon equations introduced by I. Segal [8] is used to describe the motion of charged mesons in an electromagnetic field. In [9], Zhang et al. proposed a linearized finite element method for solving two-dimensional fractional Klein-Gordon

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equations with a cubic nonlinear term. The method of time discretization which was proposed in [9] is a weighted combination of the L_{2-1_σ} formula. The Klein-Gordon equations can be solved by the finite difference method. In [10], Mehrdad Lakestani presented the finite difference methods and collocation methods to solve the nonlinear Klein-Gordon equations with quadratic and cubic nonlinearities. In the nonrelativistic regime, Zhang and Wang [11] introduced two compact difference schemes of fourth order for solving the nonlinear Klein-Gordon equations. There have been some existing works of stability and convergence analysis for finite difference or Fourier pseudo-spectral numerical schemes for nonlinear hyperbolic PDE, such as [12–15] and references therein, and some works of superconvergence analysis for nonlinear PDEs can be found, such as [16] and references therein. Conservation laws [17–20] play an important role in the theory of solutions. The numerical method that keeps physical quantities (such as energy, momentum, or mass) stable is important. R. Abgrall et al. focused on kinetic models that are BGK approximation of hyperbolic conservation laws and given an extension of the residual distribution framework to stiff relaxation problems in [27]. Deng et al. [28] considered two kinds of finite difference method of energy preserving for the coupled Klein-Gordon equations in two dimensions. The nonlinear Klein-Gordon equations are Hamiltonian wave equations and it can be written as a Hamiltonian system [35]. Nevertheless, there are a few results on the energy conserving finite element methods to solve the coupled nonlinear Klein-Gordon equations. Meanwhile, many models which is similar to the systems of coupled Klein-Gordon equations or coupled Sine-Gordon equations are applied to simulate other physical and chemical problems, for example, the long-wave limit of a lattice model for 1D nonlinear wave processes in a bilayer, the model of crack propagation in composites, the model of bi-materials, and dynamical processes in hydrogen-bonded chains, molecular crystals and polymer chains as well as ferroelectrics or ferromagnets and thin films where rotational and vibrational degrees of freedom are coupled together (see [28,32–34] and references therein). In [30,31] the inverse inequality was used to estimate the boundedness of numerical solutions, the constraint between spatial mesh size and time step was considered to obtain the optimal error estimate for the numerical scheme.

Recently, in [36–40], the error between exact solution and numerical solution is divided into two parts: temporal error and spatial error, which is different from the traditional numerical analysis, the restriction on spatial mesh size and time step can be eliminated by using the error splitting technique.

In this paper, we study the energy conserving finite element method for the coupled Klein-Gordon equations. First, the nonlinear coupled Klein-Gordon equations are rewritten into vector form, and the energy conserving numerical scheme is proposed. Second, it is shown that the proposed numerical scheme is energy-conserving. Thirdly, in order to eliminate the restriction on spatial mesh size and time step, the Crank-Nicolson discrete scheme is used in temporal approach and the bilinear finite element to is used in space approach, respectively, which implies that the proposed discrete scheme is unconditionally stable. Finally, we derive the superconvergence rate $\mathcal{O}(h^2+\Delta t^2)$ in the sense of energy normal by using the postprocessing interpolation operator. We also give some