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# An Algebraic Multigrid Method for Nearly Incompressible Elasticity Problems in Two-Dimensions

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**Abstract.** In this paper, we discuss an algebraic multigrid (AMG) method for nearly incompressible elasticity problems in two-dimensions. First, a two-level method is proposed by analyzing the relationship between the linear finite element space and the quartic finite element space. By choosing different smoothers, we obtain two types of two-level methods, namely TL-GS and TL-BGS. The theoretical analysis and numerical results show that the convergence rates of TL-GS and TL-BGS are independent of the mesh size and the Young's modulus, and the convergence of the latter is greatly improved on the order $p$. However the convergence of both methods still depends on the Poisson's ratio. To fix this, we obtain a coarse level matrix with less rigidity based on selective reduced integration (SRI) method and get some types of two-level methods by combining different smoothers. With the existing AMG method used as a solver on the first coarse level, an AMG method can be finally obtained. Numerical results show that the resulting AMG method has better efficiency for nearly incompressible elasticity problems.

**AMS subject classifications**: 65N55, 65N22
**Key words**: Locking phenomenon, algebraic multigrid, higher-order finite element, two-level method, reduced integration.

## 1 Introduction

In practice, a linear elasticity analysis is often required during simulations of multi-physics problems such as micro-electro-mechanical systems [1]. Typically, these mod-

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ern devices involve complicated geometries, extremely high aspect ratios and disparate material properties and a great number of such problems must be solved by some numerical methods, in which the finite element method is the most commonly used numerical method for their analysis [2, 3]. There are many materials in applications such as rubber and plastic, which show nearly incompressible material properties, i.e., poisson's ratio $\nu$ close to 0.5 or *Lamé* constant $\lambda$ close to $\infty$. For the planar linear elasticity, it is well known that some finite element schemes result in poor convergence in the displacements (diverge or cannot obtain the optimal order of convergence) as $\lambda$ is close to $\infty$. This is the so-called poisson's locking phenomenon in engineering. There are many literatures involving the locking phenomenon of the finite element method [6–10] and more detailed explanation of locking effects can be found in [4–6].

In order to overcome this poisson's locking, we need to construct some finite element schemes whose optimal error estimates are uniform with respect to $\lambda \in (0, \infty)$. Several approaches are developed in recent years, such as mixed finite element method based on Hellinger-Reissner variational principle [5, 7, 11–14], nonconforming finite element method [15–19], $p$-version and $hp$-version method of higher-order finite element scheme [4, 20–23], selective reduced integration method [24] that is often equivalent to a mixed method, and etc. Since discrete variation formulas, based on the minimization of the energy functional, are easier to be solved than the mixed formula, we consider this formula with pure displacement boundary condition. Moreover, the system matrix is positive definite, such that the CG method can be applied. In addition, while nonconforming finite elements has few degree of freedom, this method is sensitive to the adopted mesh. Therefore, in this paper we consider higher-order conforming finite element scheme to overcome this locking.

In 1983, M. Vogelius [20] considered conforming finite element approximations to the linear planar elasticity as $\lambda$ is close to $\infty$, and showed that the piecewise linear conforming finite element scheme did not converge any more, and the quadratic and cubic conforming finite element schemes could not obtain optimal error estimates. In [21], it was shown that no locking results could be obtained when polynomials of degree $p \geq 4$ on a triangular mesh. Hence, $p$-version and $hp$-version method of higher-order ($p \geq 4$) conforming finite element scheme is an important approach to overcome the poisson's locking [23]. However, they have much higher computational complexity than the low-order elements, and the system matrix is often large-scale, symmetric and positive definite and ill-conditioned.

Algebraic multigrid (AMG) method for system of PDEs, such as the equations of linear elasticity, seems to be premature and the naive use of the scalar AMG does not lead to the robust and efficient solver. We would like to refer readers to [25–29] for the recent efforts to apply AMG methods for system of PDEs. But these methods are just for linear discretizations. For the higher-order discretization of system of PDEs, there are few studies on designing fast solvers. Recently, several efficient AMG methods for $\nu \leq 0.4$ are developed in [30] for higher-order discretizations by using geometric and algebraic information. Theoretically, it can be viewed as a two-level method proposed