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An Efficient Second-Order Finite Volume ADI Method for Nonlinear Three-Dimensional Space-Fractional Reaction-Diffusion Equations

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Abstract. In this paper, a three-dimensional time-dependent nonlinear Riesz space-fractional reaction-diffusion equation is considered. First, a linearized finite volume method, named BDF-FV, is developed and analyzed via the discrete energy method, in which the space-fractional derivative is discretized by the finite volume element method and the time derivative is treated by the backward differentiation formulae (BDF). The method is rigorously proved to be convergent with second-order accuracy both in time and space with respect to the discrete and continuous L^2 norms. Next, by adding high-order perturbation terms in time to the BDF-FV scheme, an alternating direction implicit linear finite volume scheme, denoted as BDF-FV-ADI, is proposed. Convergence with second-order accuracy is also strictly proved under a rough temporal-spatial stepsize constraint. Besides, efficient implementation of the ADI method is briefly discussed, based on a fast conjugate gradient (FCG) solver for the resulting symmetric positive definite linear algebraic systems. Numerical experiments are presented to support the theoretical analysis and demonstrate the effectiveness and efficiency of the method for large-scale modeling and simulations.

AMS subject classifications: 65M08, 65M12, 65M15

Key words: Nonlinear space-fractional reaction-diffusion equation, finite volume method, ADI, convergence, efficient implementation.

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1 Introduction

In the past decades, nonlinear space-fractional differential equations (s-FDEs) have been shown to provide an adequate and accurate description for challenging phenomena such as long-range interaction and anomalously diffusive transport in various science and engineering fields. For example, the fractional Allen-Cahn equation [2, 12] was applied to describe the mesoscale morphological pattern formation and interface motion; the fractional FitzHugh-Nagumo model [3] was used to represent impulse propagation in nerve membranes; and the fractional Bloch-Torrey equation [21] has been successfully used in magnetic resonance. However, in most case it is not available to obtain the analytical solutions [15,32] for fractional differential equations. Therefore, efficient numerical modeling becomes extremely urgent and important. Up to now, there has been an increasing interest in developing and analyzing efficient numerical methods, see [4, 7, 16, 19, 20, 23, 24, 28, 31, 35, 37, 38] and the references therein.

Due to the local conservation property, the finite volume (FV) method is particularly suitable for modeling and simulation of conservative type s-FDEs. Hejazi and Moroney [10] presented a finite volume approximation to the one-dimensional time-space fractional advection-dispersion equation, and showed that this method performs better than finite difference method for the considered problem with variable coefficient, since it deals with the equation directly in a conservative form. A preconditioned Lanczos method which uses finite volume spatial discretization for space-fractional reaction-diffusion equations was proposed and verified to be suitable for unstructured meshes in [36]. Liu et al. [20] presented a finite volume method for the space-fractional diffusion equation with variable coefficients and nonlinear source term. Simmons and Yang [27] developed a novel finite volume discretization based on non-uniform meshes for two-sided fractional diffusion equations with Riemann-Liouville derivative and proved the stability of the scheme. In order to obtain second-order temporal accuracy, some numerical techniques like Crank-Nicolson method [8, 38] and backward differentiation formulae (BDF) [5, 13] were considered for related fractional models. In particular, Fu et al. presented second-order Crank-Nicolson FV approximations for the two-dimensional s-FDEs [8], and for the three-dimensional nonlinear distributed-order s-FDEs [41]. Corresponding unconditional stability and error estimates in discrete energy norms were rigorously studied. However, the FV scheme coupling with the BDF method for nonlinear space-fractional models has not been studied yet.

In this paper, we are interested in the following three-dimensional nonlinear Riesz space-fractional reaction-diffusion equation (s-FRDEs) with orders α $(1<\alpha<2)$ in x- direction, β $(1<\beta<2)$ in y- direction and γ $(1<\gamma<2)$ in z-direction [6, 13]:

$$\frac{\partial u}{\partial t}-d_x\frac{\partial^\alpha u(\mathbf{x},t)}{\partial|x|^\alpha}-d_y\frac{\partial^\beta u(\mathbf{x},t)}{\partial|y|^\beta}-d_z\frac{\partial^\gamma u(\mathbf{x},t)}{\partial|z|^\gamma}=f(u)+g(\mathbf{x},t),\quad (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.1a}$$

$$u(\mathbf{x},t)=0,\quad (\mathbf{x},t)\in\partial\Omega\times[0,T],\quad u(\mathbf{x},0)=u^0(\mathbf{x}),\qquad \mathbf{x}\in\Omega, \tag{1.1b}$$

where $T<\infty$ is the final time instant, $\partial\Omega$ is the boundary of $\Omega\subset\mathbb{R}^3$ and $\mathbf{x}=(x,y,z)$.