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A Well-Balanced Runge-Kutta Discontinuous Galerkin Method for Multilayer Shallow Water Equations with Non-Flat Bottom Topography

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Abstract. A well-balanced Runge-Kutta discontinuous Galerkin method is presented for the numerical solution of multilayer shallow water equations with mass exchange and non-flat bottom topography. The governing equations are reformulated as a non-linear system of conservation laws with differential source forces and reaction terms. Coupling between the flow layers is accounted for in the system using a set of exchange relations. The considered well-balanced Runge-Kutta discontinuous Galerkin method is a locally conservative finite element method whose approximate solutions are discontinuous across the inter-element boundaries. The well-balanced property is achieved using a special discretization of source terms that depends on the nature of hydrostatic solutions along with the Gauss-Lobatto-Legendre nodes for the quadrature used in the approximation of source terms. The method can also be viewed as a high-order version of upwind finite volume solvers and it offers attractive features for the numerical solution of conservation laws for which standard finite element methods fail. To deal with the source terms we also implement a high-order splitting operator for the time integration. The accuracy of the proposed Runge-Kutta discontinuous Galerkin method is examined for several examples of multilayer free-surface flows over both flat and non-flat beds. The performance of the method is also demonstrated by comparing the results obtained using the proposed method to those obtained using the incompressible hydrostatic Navier-Stokes equations and a well-established kinetic method. The proposed method is also applied to solve a recirculation flow problem in the Strait of Gibraltar.

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1 Introduction

The incompressible Navier-Stokes equations have been well established as accurate tools for modelling and simulating water flows, see [10, 47] among others. However, for free-surface flows, these equations become difficult to numerically solve mainly due to the hydrostatic assumption on the pressure and the presence of moving boundaries within the flow domain. On the other hand, assuming that the pressure is hydrostatic and the vertical scale is far smaller than the horizontal scale, the shallow water equations can be derived by depth-averaging the three-dimensional Navier-Stokes equations, see for instance [1]. Indeed, shallow water equations have been widely used in modelling many engineering applications in free-surface flows and hydraulics such as tides in coastal regions, floods in rivers, water flows in reservoir, and open channel flows among others, see for example [33, 43]. However, because these equations are derived based on depth-averaged procedures, the vertical velocity component is not resolved in these models and the bed friction is derived only in terms of the mean flow velocity rather than the velocity near the bottom. Hence, the three-dimensional modeling of hydraulics is required for an accurate representation of the flow structures, especially for recirculation flows and for solution of near-field problems involving sediment transport in rivers and coastal engineering. In the recent years, research in shallow water flows has been shifted to overcome the shortcomings of this type of modeling namely, the use of single velocity profile for the entire depth of the water flow. This has conducted to the introduction of multilayer shallow water equations for geophysical free-surface flows. For example, two-layer shallow water equations have been proposed to model immiscible fluids in [16, 24]. More recently, multilayer shallow water equations with mass exchange terms have been studied in [4, 6, 11, 13] among others. The governing equations in these multilayer models have been derived using a P_0 finite element discretization of the vertical velocity in the three-dimensional incompressible Navier-Stokes equations. This class of multilayer shallow water equations avoid the computationally demanding methods needed to solve the three-dimensional incompressible Navier-Stokes equations with free-surface but at the same time providing stratified flow velocities since the pressure distribution is still assumed to be hydrostatic. In the current study, the free-surface flow problem is approximated as a layered system made of multiple shallow water systems of different water heights but coupled through mass-exchange terms between the embedded layers. These water layers may also differ in terms of density, viscosity, compressibility and potential for mixing among others. Recently, multilayer shallow water equations have been subject of various research studies and have been used for modelling a variety of free-surface flows such as estuaries, bays and other nearshore regions where water free-surface flows