Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **14**, No. 3, pp. 666-702

DOI: 10.4208/aamm.OA-2020-0237
June 2022

A Numerical-Analytical Method for Time-Fractional Dual-Phase-Lag Models of Heat Transfer

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Received 2 August 2020; Accepted (in revised version) 22 December 2020

Abstract. The aim of this paper is to present the backward substitution method for solving a class of fractional dual-phase-lag models of heat transfer. The proposed method is based on the Fourier series expansion along the spatial coordinate over the orthonormal basis formed by the eigenfunctions of the corresponding Sturm-Liouville problem. This Fourier expansion of the solution transforms the original fractional partial differential equation into a sequence of multi-term fractional ordinary differential equations. These fractional equations are solved by the use of the backward substitution method. The numerical examples with temperature-jump boundary condition and parameters of the tissue confirm the high accuracy and efficiency of the proposed numerical scheme.

AMS subject classifications: 65N35, 80A20

Key words: Heat transfer, dual-phase-lag model, fractional partial differential equation, semi-analytical method.

1 Introduction

Fractional partial differential equations (FPDEs) are widely used in various areas of science and engineering. Their advantages become apparent in modeling electrical properties of real materials, the so-called anomalous transport phenomena, and in the theory of complex systems. The FPDEs describe important physical phenomena that arise in

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amorphous, colloid, glassy and porous materials, in fractals and percolation clusters, dielectrics and semiconductors, biological systems, polymers, random and disordered media, geophysical and geological processes [1–10].

Recently there has been a growing interest for fractional bioheat dual-phase-lag (DPL) heat transfer models [11–16, 18–23] in order to get a precise prediction of thermal data within living biological tissues in different thermal treatment processes.

In this paper we present a novel method for the class of FPDEs

$$\mathcal{L}_t[v]=\mathcal{M}_t\left\{\frac{\partial^2 v(x,t)}{\partial x^2}\right\}-p^2\mathcal{P}_t\{v(x,t)\}+f(x,t),\quad 0\le x\le 1,\quad 0\le t\le T, \tag{1.1}$$

where

$$\mathcal{L}_t=D_t^{(\mu)}+\sum_{k=1}^{I}\alpha_k D_t^{(\mu_k)},\quad \mathcal{M}_t=\sum_{k=I+1}^{J}\alpha_k D_t^{(\mu_k)},\quad \mathcal{P}_t=\sum_{k=J+1}^{K}\alpha_k D_t^{(\mu_k)}, \tag{1.2}$$

are time fractional differential operators and $\mu\in(l-1,l]$ is the order of the higher derivative of the differential operator $\mathcal{L}_t$. The integer number l defines the maximal value of the fractional order μ and so, the amount of the initial conditions needed for the FPDE. The values $0\le\mu_k<\mu$ are fractional or integer constant numbers, $p\ge 0$ and α_k, $k=1,\cdots,K$, are real numbers.

Eq. (1.1) is subjected to the initial conditions (ICs)

$$v(x,0)=v_0(x),\quad \frac{\partial v(x,0)}{\partial t}=v_1(x),\cdots,\frac{\partial^{l-1}v(x,0)}{\partial t^{l-1}}=v_{l-1}(x). \tag{1.3}$$

We utilize $l=2$ and $l=3$ for DPL models of the first and second order respectively.

We write the boundary conditions (BCs) along the spatial coordinate in the general form

$$\mathcal{B}_0[v]_{x=0}=g_0(t),\quad \mathcal{B}_1[v]_{x=1}=g_1(t), \tag{1.4}$$

where the operators $\mathcal{B}_0$, $\mathcal{B}_1$ conform Dirichlet's, Neumann's or Robin's conditions.

The operator $D_t^{(\nu)}$ denotes the Caputo fractional derivative defined by [26]

$$D_t^{(\nu)}[T(x,t)]=\begin{cases}\dfrac{1}{\Gamma(n-\nu)}\displaystyle\int_0^x\frac{T^{(n)}(x,\tau)d\tau}{(t-\tau)^{\nu-n+1}}, & n-1<\nu<n,\\ T^{(n)}(x,t), & \nu=n,\end{cases} \tag{1.5}$$

where $n\in\mathcal{N}=\{1,2,\cdots\}$ is the set of positive integers, and $\Gamma(z)$ denotes the gamma function. In particular, for the power functions we get:

$$D_t^{(\nu)}[t^z]=\begin{cases}0, & \text{if } z\in\mathcal{N}_0 \text{ and } z<n,\\ \dfrac{\Gamma(z+1)}{\Gamma(z+1-\nu)}t^{z-\nu}, & \text{if } z\in\mathcal{N}_0 \text{ and } z\ge n \text{ or } z\notin\mathcal{N}_0 \text{ and } z>n-1,\end{cases} \tag{1.6}$$