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A Posteriori Error Estimates for hp Spectral Element Approximation of Elliptic Control Problems with Integral Control and State Constraints

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Abstract. This paper investigates an optimal control problem governed by an elliptic equation with integral control and state constraints. The control problem is approximated by the hp spectral element method with high accuracy and geometric flexibility. Optimality conditions of the continuous and discrete optimal control problems are presented, respectively. The a posteriori error estimates both for the control and state variables are established in detail. In addition, illustrative numerical examples are carried out to demonstrate the accuracy of theoretical results and the validity of the proposed method.

AMS subject classifications: 35Q93, 49M25, 49M41

Key words: Elliptic equations, optimal control, control-state constraints, a posteriori error estimates, hp spectral element method.

1 Introduction

Optimal control problems have been widely applied in many fields, such as physical, manufacturing, ecological environment, medical treatment, social, and economic processes, etc. We point the reader to the classical works [27, 33, 49] for optimal control problems in detail. In various classes of optimal control problems of the PDE-constraints

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type the finite element method has been applied to solve the problems, see e.g., [2, 6, 16–19, 22, 23, 27–31, 34–37, 40, 43, 55]. In [6], the authors considered finite element approximation of a family of elliptic optimal control problems with pointwise constraints on control and state, and derived the discretization error for a problem with mixed state constraints in the semidiscrete case and in the fully discrete scheme, respectively. An elliptic optimal control problem with control and pointwise state constraints were investigated by the finite element method in [18], and the error bounds for control and state were obtained both in two and three space dimensions. A discussion of numerical approximation to a parabolic control problem with control and state constraints has been studied in [23], the standard piecewise linear and continuous finite elements and the dG(0) method were used for space and time discretization, respectively, and a priori error estimates for control and state were obtained. [31] considered a state-constrained optimal control problem with boundary control, where the state constraints are imposed only in an interior subdomain, and derived a priori error estimates for a finite element discretization with and without additional regularization. In [43], the authors discussed the full discretization of an elliptic optimal control problem with pointwise control and state constraints, provided the first reliable a-posteriori error estimator which tends to zero if the solution of the discretized problems converges to the solution of the undiscretized problem.

The spectral method, aiming to employ the global polynomials as the trial functions for the discretization of PDEs to achieve the fast convergence rate and high-order accuracy, has been successfully applied in numerical solutions for partial differential equations (PDEs) (see e.g., [3, 15, 21, 25, 26, 42, 45–48, 50–54]) and becomes nowadays a popular approach in solving the optimal control problems governed by PDEs in [7–10, 38, 56–59]. The authors studied in [7] the Legendre-Galerkin spectral approximation for a constrained optimal control problem governed by an elliptic equation, a priori and a posteriori error estimates for the spectral approximation of optimal control problems were derived, and the a posteriori estimates were demonstrated for the hp spectral element method. The Legendre-Galerkin spectral method for both the unconstrained and the constrained optimal control problems governed by Stokes equations were investigated, and the error estimates were established in [8]. A Galerkin spectral method was discussed for an optimal control problem governed by the time fractional diffusion equation with constraints on the control variable in [56], a priori error estimates for the space-time spectral approximation were derived, and a projection gradient algorithm was designed to solve the discrete minimization problem. In [59], the Legendre Galerkin spectral approximations of an integral constraint on state optimal control problem governed by the first bi-harmonic equation were constructed, a priori error estimates of high accuracy were obtained, and an projection algorithm was proposed.

The hp-version method, which allowing for locally varying steps and approximation orders to enhance the numerical accuracy significantly, have been proposed and analyzed for practical problems. The pioneering work is studied by Babuška [1]. Recently, some efforts have been put into developing hp-version methods for solving optimal control problems [11–13, 24, 39]. A posteriori error analysis for hp finite element approximation