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Singularity of Navier-Stokes Equations Leading to Turbulence

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Abstract. Singularity of Navier-Stokes equations is uncovered for the first time which explains the mechanism of transition of a smooth laminar flow to turbulence. It is found that when an inflection point is formed on the velocity profile in pressure driven flows, velocity discontinuity occurs at this point. Meanwhile, pressure pulse is produced at the discontinuity due to conservation of the total mechanical energy. This discontinuity makes the Navier-Stokes equations be singular and causes the flow to become indefinite. The analytical results show that the singularity of the Navier-Stokes equations is the cause of turbulent transition and the inherent mechanism of sustenance of fully developed turbulence. Since the velocity is not differentiable at the singularity, there exist no smooth and physically reasonable solutions of Navier-Stokes equations at high Reynolds number (beyond laminar flow). The negative spike of velocity and the pulse of pressure due to discontinuity have obtained agreement with experiments and simulations in literature qualitatively.

AMS subject classifications: 76D03, 76D05, 76E05, 76F06

Key words: Navier-Stokes equations, singularity, discontinuity, total mechanical energy, turbulence.

1 Introduction

Turbulence is one of the most important scientific problems in physics. Reynolds pioneered the work for pipe flow in 1883, which proved that there are two types of flow states, laminar flow and turbulence [1]. Since then, substantial work has been done in theories, experiments and simulations on turbulence during the past 135 years or so. Although great progress has been made, the physical mechanism of turbulence is still poorly understood.

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Over the years, it has been suggested that transition of laminar flow to turbulence is caused by the instability of laminar flow [2,3]. The flow between two parallel plates is a classical flow problem. Heisenberg [2] obtained an approximate solution of the stability equation by the derivation of the linearized Navier-Stokes equations and gave the boundary of stability on the diagram of wave number versus the Re number. Lin [3] proved by mathematical asymptotic analysis that the flow between the two parallel plates would be unstable and obtained the critical Re number to be 8000. However, whether or not turbulence would occur after flow instability setting in was not clarified. Later, several researchers obtained similar results which approach the accurate value gradually. Orszag obtained the critical Re number to be 5772 by calculating with the spectral method [4]. This is the most accurate value recognized in the literature. However, the critical Re number of turbulent transition obtained from experiments is about 1000. This inconsistency between the theory and the experiment has been perplexing the understanding of turbulent transition [4–7].

A large quantity of experimental data and numerical calculations show that turbulence is a local phenomenon when it first starts [8–14]. With the increase of Re and the development of disturbance, inflection point first appears on the velocity profile in laminar flow, followed by the formation of hairpin vortices which finally leads to the turbulence spots. During the transition, the velocity profile is subjected to a continuous modification from a laminar profile to a turbulent profile [7, 11, 14–18]. It is found that the appearance of the velocity inflection point is a key step of turbulent transition [11–18] and recently experiment confirmed that turbulence is indeed sustained by an inflection point instability [13].

In the experiment of boundary layer flow [9], it was discovered that breakdown of the laminar flow starts with the appearance of negative spikes on the streamwise velocity temporal traces under influence of disturbances. The evolution is from one spike, then the spike doubled, tripled, etc. downstream. In some experiment and simulation, it was shown that the maximum amplitude of the negative spike reaches more than 50% of the streamwise velocity [19]. Kachanov reviewed the progress of researches on transition in boundary layer flow, and discussed the transition phenomenon and the roles of spikes in the flow breakdown [20]. However, the origin of the spike is still not fully understood up to present.

In the experiments and simulations, it is also found that negative spikes of streamwise velocity appear during the late stage of the transition of plane Poiseuille flow or channel flows [17,21–24]. The simulation results showed that the appearance of the negative spike in streamwise velocity is accompanied by the distortion/inflection of velocity profile and the high-shear layer sloped to the wall [17, 18].

Dou and co-authors proposed an energy gradient theory for the study of flow stability and turbulent transition [25–34]. It is found that the inflection point on the velocity profile for the pressure driven flow is a singular point hidden in the Navier-Stokes equations. It is obtained for pressure driven flows that the necessary and sufficient condition for the turbulent transition is the existence of an inflection point on the velocity profile [25, 27,