Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 5, pp. 1196-1223

DOI: 10.4208/aamm.OA-2019-0256
October 2020

An Upwind Mixed Finite Element Method on Changing Meshes for Positive Semi-Definite Oil-Water Displacement of Darcy-Forchheimer Flow in Porous Media

Yirang Yuan[1,*], Huailing Song[2], Changfeng Li[1,3,*] and Tongjun Sun[1]

[1] *Institute of Mathematics, Shandong University, Jinan, Shandong 250100, China*
[2] *College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, China*
[3] *School of Economics, Shandong University, Jinan, Shandong 250100, China*

Received 25 August 2019; Accepted (in revised version) 17 March 2020

Abstract. An upwind mixed finite element method is proposed on changing meshes for solving a positive semi-definite miscible displacement problem of Darcy-Forchheimer flow in three-dimensional porous media. The pressure and velocity could be obtained together by using a mixed finite element, and the computational accuracy of velocity is improved. The concentration equation is solved by the upwind mixed finite element scheme on changing meshes, where the upwind approximation and an expanded mixed finite element are adopted for the convection and diffusion, respectively. It solves the convection-dominated diffusion problem well and has the following improvements. First, the conservation of mass, an important physical nature, is preserved. Second, it has high order computational accuracy. An optimal-order error estimates is concluded. Numerical experiments illustrate the efficiency and application of the presented scheme.

AMS subject classifications: 65N30, 65N12, 65M15, 65M60

Key words: Darcy-Forchheimer flow, positive semi-definite problem, adaptive changing meshes, upwind mixed finite element method, convergence analysis.

1 Introduction

The pressure-velocity mathematical model is generally used for interpreting the Darcy-Forchheimer's law, and describes the miscible displacement of two-phase flow in porous

*Corresponding author.
Emails: yryuan@sdu.edu.cn (Y. R. Yuan), cfli@sdu.edu.cn (C. F. Li)

media. A nonlinear partial differential system with initial-boundary conditions is formulated for the incompressible displacement [1–4].

$$\mu(c)\kappa^{-1}\mathbf{u}+\beta\rho(c)|\mathbf{u}|\mathbf{u}+\nabla p=r(c)\nabla d, \qquad X=(x,y,z)^T\in\Omega, \qquad t\in J=(0,\bar{T}], \tag{1.1a}$$

$$\nabla\cdot\mathbf{u}=q=q_I+q_p, \qquad X\in\Omega, \qquad t\in J, \tag{1.1b}$$

$$\phi\frac{\partial c}{\partial t}+\mathbf{u}\cdot\nabla c-\nabla\cdot(D(\mathbf{u})\nabla c)+q_Ic=q_Ic_I, \qquad X\in\Omega, \qquad t\in J=(0,T], \tag{1.1c}$$

where Ω denotes a bounded three-dimensional domain.

To describe the displacement of highspeed flow in heterogeneous media especially nearby the wells [2], Forchheimer put forward one mathematical model of Darcy-Forchheimer flow and discussed its numerical simulation in [1]. Darcy-Forchheimer's law is Darcy's law when $\beta=0$. The law of Darcy-Forchheimer was discussed in [3], and the regularity was analyzed in [4].

The problem of (1.1) and (1.1c) satisfy the conservative law of mass. Functions $p(X,t)$, $\mathbf{u}(X,t)$ and $c(X,t)$ denote the pressure, Darcy velocity and the concentration of one component, respectively. The coefficients are interpreted as follows, $\kappa(X)$, the absolute permeability, $\phi(X)$, the porosity, $\beta(X)$, Forchheimer coefficient, $r(X,c)$, the gravity coefficient, and $d(X)$, the vertical coordinate. $q(X,t)$, the production, is usually defined by a linear function of the production q_p and the injection q_I, $q(X,t)=q_I(X,t)+q_p(X,t)$. The externally-imposed concentration at an injection well, c_I, is known.

Suppose that two fluids are incompressible and their total volume does not decrease. Suppose that no chemical reaction takes place. Let ρ_1 and ρ_2 denote the densities of different fluids and let the density of their mixture be denoted by

$$\rho(c)=c\rho_1+(1-c)\rho_2. \tag{1.2}$$

The mixture's viscosity is determined by

$$\mu(c)=\left(c\mu_1^{-1/4}+(1-c)\mu_2^{-1/4}\right)^{-4}. \tag{1.3}$$

$D(\mathbf{u})$ is the diffusion tensor of molecular diffusion and mechanical dispersion, generally defined by

$$\begin{aligned}\mathbf{D}(X,\mathbf{u})=&\phi d_m\mathbf{I}+d_l|\mathbf{u}|^\beta\begin{pmatrix}\hat{u}_x^2 & \hat{u}_x\hat{u}_y & \hat{u}_x\hat{u}_z\\ \hat{u}_x\hat{u}_y & \hat{u}_y^2 & \hat{u}_y\hat{u}_z\\ \hat{u}_x\hat{u}_z & \hat{u}_y\hat{u}_z & \hat{u}_z^2\end{pmatrix}\\ &+d_t|\mathbf{u}|^\beta\begin{pmatrix}\hat{u}_y^2+\hat{u}_z^2 & -\hat{u}_x\hat{u}_y & -\hat{u}_x\hat{u}_z\\ -\hat{u}_x\hat{u}_y & \hat{u}_x^2+\hat{u}_z^2 & -\hat{u}_y\hat{u}_z\\ -\hat{u}_x\hat{u}_z & -\hat{u}_y\hat{u}_z & \hat{u}_x^2+\hat{u}_y^2\end{pmatrix},\end{aligned} \tag{1.4}$$

where d_m is the molecular diffusivity, $\mathbf{I}$ is a 3×3 identity matrix, d_l and d_t are the longitudinal and transverse dispersivities, respectively, and $\hat{u}_x$, $\hat{u}_y$ and $\hat{u}_z$ represent the three