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Unconditional Long Time H^1-Stability of a Velocity-Vorticity-Temperature Scheme for the 2D-Boussinesq System

Mine Akbas*

Department of Mathematics, Duzce University, Duzce 81620, Turkey

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Abstract. This paper proposes, analyzes and tests a velocity-vorticity-temperature (VVT) scheme for incompressible, non-isothermal fluid flow. VVT consists of complementing of the usual velocity-pressure-temperature system with the vorticity equation, coupling the systems through the convective terms. The proposed scheme uses BDF2LE time stepping, and a finite element spatial discretization. At each time step, the velocity-pressure equation, the vorticity equation and the temperature equation are all decoupled. A full analysis of the method is given that proves unconditional long-time H^1-stability, and shows the optimal convergence both in time and space. Theoretical convergence results are confirmed by a numerical test, and the effectiveness of the algorithm is revealed on a benchmark problem for Marsigli flow.

AMS subject classifications: 74H40, 76D03, 76D05, 65N30

Key words: Long time stability, incompressible flow, vorticity equation, finite element method.

1 Introduction

Incompressible, non-isothermal fluid flows are governed by the incompressible Navier-Stokes equations (NSE) and heat transport equation, and read as: for a given force field $\mathbf{f}:[0,T]\times\Omega\to\mathbb{R}^d$, find a velocity field $\mathbf{u}:[0,T]\times\Omega\to\mathbb{R}^d$, and pressure $p:(0,T]\times\Omega\to\mathbb{R}$ and temperature fields $\theta:[0,T]\times\Omega\to\mathbb{R}$ such that $(\mathbf{u},p,\theta)$ satisfies the equations

$$\frac{\partial \mathbf{u}}{\partial t}-\nu\Delta\mathbf{u}+\boldsymbol{u}\cdot\nabla\mathbf{u}+\nabla p=Ri\theta\boldsymbol{\xi}+\mathbf{f} \qquad \text{in } (0,T]\times\Omega, \tag{1.1a}$$

$$\nabla\cdot\mathbf{u}=0 \qquad \text{in } (0,T]\times\Omega, \tag{1.1b}$$

$$\frac{\partial \theta}{\partial t}-\kappa\Delta\theta+(\mathbf{u}\cdot\nabla)\theta=\gamma \qquad \text{in } (0,T]\times\Omega, \tag{1.1c}$$

*Corresponding author.
Email: mineakbas@duzce.edu.tr (M. Akbas)

with appropriate boundary and initial conditions. The problem is posed on a bounded domain with Lipschitz continuous boundary. Here, $\boldsymbol{\xi}$ denotes the unit vector in the direction of gravity, $\nu =: 1/Re$ is the dimensionless kinematic viscosity, where Re denotes the Reynolds number, $Ri =: Gr/Re^2$ is the Richardson number which accounts for the gravitational force and the thermal expansion of the fluid, and $\kappa =: 1/(PrRe)$ is thermal diffusivity coefficient.

In numerical simulations of the incompressible flow problems, different formulations such as the convective, rotational, conservative, EMAC, and vorticity-stream, vorticity-helicity can be found in the literature. Even though these formulations are equivalent at continuous level, they can lead to very different results on specific problems when discretized [5, 13, 23, 26]. Among them, the velocity-vorticity (VV) formulation has been shown to be an efficient approach to simulate incompressible and compressible flows, [4, 6, 7, 9–11, 14, 16, 26]. VV formulations were first derived by Fasel [11] to study the stability of boundary layers in two dimensions, and later extended to steady-state three-dimensional incompressible flows [8]. Such formulations are very useful for a wide range of incompressible flows, both in two and three dimensions [2, 18, 24, 26, 31, 32, 34], in particular for vortex dominated or strongly rotating flows. This is because the vorticity advection term in these flows has a fundamental importance in determining the flow dynamics, and vorticity is a primary variable (not determined by taking derivative of a primary variable). Besides its advantages, a limitation of VV formulations is the lack of mathematical analysis for three dimensional, time-dependent incompressible flows. This is due to the complexity of the estimating vortex stretching term, which is the critical difference between the vorticity equation in 2D and 3D. Thus, such formulations are often analyzed in the 2D setting, in order to gain insight into 3D behavior.

In the very recent papers [18] and [2], a particular VV formulation for 2D incompresible NSE was studied with linearly extrapolated BDF2 (BDF2LE) time stepping and finite element spatial discretization. In [18], it was proven that these methods' velocity and vorticity solutions are unconditionally long-time H^1-stable. The novelty of the stability analysis in [18] is that the stability bounds are not subject to any time step restriction, which is very common in the topic of long time stability [3, 12, 20, 21, 29, 30, 33], and estimates do not have exponential dependence on Re.

The purpose of this paper is to study a special velocity-vorticity-temperature (VVT) formulation and numerical scheme for the incompressible Boussinesq equations in 2D, and to extend the results of [2, 18] to this case. The VVT formulation for 2D-Boussinesq equations are constructed by coupling the vorticity equation to (1.1) by means of the rotational form of the non-linearity in the momentum equation, which is given below

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + w\times\mathbf{u} + \nabla P = Ri\theta\boldsymbol{\xi} + \mathbf{f} \qquad \text{in } (0,T]\times\Omega, \tag{1.2a}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } (0,T]\times\Omega, \tag{1.2b}$$

$$\frac{\partial \theta}{\partial t} - \kappa\Delta\theta + (\mathbf{u}\cdot\nabla)\theta = \gamma \qquad \text{in } (0,T]\times\Omega, \tag{1.2c}$$