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A Stabilized Finite Element Method for Non-Stationary Conduction-Convection Problems

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Abstract. This paper is concerned with a stabilized finite element method based on two local Gauss integrations for the two-dimensional non-stationary conduction-convection equations by using the lowest equal-order pairs of finite elements. This method only offsets the discrete pressure space by the residual of the simple and symmetry term at element level in order to circumvent the inf-sup condition. The stability of the discrete scheme is derived under some regularity assumptions. Optimal error estimates are obtained by applying the standard Galerkin techniques. Finally, the numerical illustrations agree completely with the theoretical expectations.

AMS subject classifications: 65N30, 65N12, 65Z05

Key words: Non-stationary conduction-convection equations, finite element method, stabilized method, stability analysis, error estimate.

1 Introduction

The non-stationary conduction-convection problems are the important dissipative nonlinear system in atmospheric dynamics. The governing equations couple viscous incompressible flow and heat transfer process [10], where the incompressible fluid is the Boussinesq approximation to the non-stationary Navier-Stokes equations. Christon [5] summarized some relevant results for the fluid dynamics of thermally driven cavity. A multigrid (MG) technique was applied for the conduction-convection problems [23, 24]. Luo [22] combined proper orthogonal decomposition (POD) with the Petrov-Galerkin least squares mixed finite element (PLSMFE) method for the problems. The mixed finite element (MFE) method is one of the important approaches for

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solving the non-stationary conduction-convection problems, where the combination of finite element subspaces satisfies the Babuska-Brezzi (BB) inequality [16, 26, 27]. It is well known that the simplest equal-order elements like the $P_1 - P_1$ triangular elements are not stable. However, the $P_1 - P_1$ elements are computationally convenient in a parallel processing and multigrid context with the simple logic and regular data structure.

Numerical experiments show that in the solution of the Stokes and Navier-Stokes problems, ensuring stability is essential if a reasonable rate of convergence of such iterations is to be achieved [19]. Some techniques that have been used to stabilize both the velocity and pressure belong to a class of residual-based methods. For example, the streaming upwind Petrov-Galerkin (SUPG) method [18], the Douglas-Wang method [11] and the well-known Galerkin least squares (GLS) method [12–14]. A common drawback in these stabilization techniques is, however, that stabilization parameters are necessarily incurred either explicitly or implicitly. Thus the development of mixed finite elements free from the stabilization parameters has become increasingly important. Stabilized mixed finite element methods are developed by using the pressure gradient projection (PGP) method [7–9] and the related local pressure gradient stabilization (LPS) method [1] in which the continuity equation is relaxed using the jumps of pressure across element interfaces. This stabilization strategy requires edge-based data structure and a subdivision of grids into patches.

Recently, based on polynomial pressure projection, a new family of stabilized methods has been proposed and studied in [2, 3]. The new method relaxes the continuity equation to enforce the BB condition in incompatible mixed spaces by using two local Gauss integral approximation [20, 21]. It does not require a specification of mesh-dependent parameters and edge-based data structure, and it always leads to symmetric problems. In addition, it is completely local at the element level. Consequently, the new stabilized method under consideration can be integrated in existing codes with very little additional coding effort.

In this paper, we extend the stabilized finite element method [21] to the non-stationary conduction-convection equations. Firstly, we use two local Gauss integral approximation in stabilized method for the lowest equal-order pairs of mixed finite elements, such as

$$P_1 - P_1 - P_1 \quad \text{or} \quad Q_1 - Q_1 - Q_1.$$

Then we derive the stability and the optimal error estimates. The results indicate that this method has the same convergence order as the usual Galerkin finite element method using the same pair of finite elements. Finally, we numerically compare this new method with other numerical method. Numerical results show that the new method takes less CPU time than the $P_1b - P_1b - P_1$ method. The numerical illustrations agree completely with the theoretical expectations.

The remainder of the paper is organized as follows. In the next section, abstract functional setting for the two-dimensional non-stationary conduction-convection problems is given with some basic statement. The stabilized finite element method