Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 2, pp. 436-479

DOI: 10.4208/aamm.OA-2017-0243
April 2020

High-Order Finite Difference Schemes Based on Symmetric Conservative Metric Method: Decomposition, Geometric Meaning and Connection with Finite Volume Schemes

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Received 27 September 2017; Accepted (in revised version) 9 April 2019

Abstract. High-order finite difference schemes (FDSs) based on symmetric conservative metric method (SCMM) are investigated. Firstly, the decomposition and geometric meaning of the discrete metrics and Jacobian based on SCMM are proposed. Then, high-order central FDS based on SCMM is proved to be a weighted summation of second-order finite difference schemes (FDSs). Each second-order FDS has the same vectorized surfaces and cell volume as a second-order finite volume scheme (FVS), and the cell volume is uniquely determined by the vectorized surfaces. Moreover, the decomposition and connection with FVSs are also discussed for general high-order FDSs. SCMM can be applied for high-order weighted compact nonlinear scheme (WCNS). Numerical experiments show superiority of high-order WCNS based on SCMM in stability, accuracy and ability to compute flows around complex geometries. The results in this paper may to some extent explain why high-order FDSs based on SCMM can solve problems with complex geometries and may give some guidance in constructing high-order FDSs on curvilinear coordinates.

AMS subject classifications: 65D18, 68U05, 68U07

Key words: High-order, finite difference schemes, symmetric conservative metric method, finite volume schemes, complex geometries.

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1 Introduction

High-order finite difference schemes (FDSs), which can be constructed easily and have high computational efficiency, are widely used in large eddy simulations (LES) and direct numerical simulations (DNS) of turbulences and aeroacoustics [1–4]. For practical engineering problems, FDSs usually require a grid transformation from Cartesian coordinates to curvilinear coordinates. It was demonstrated that a geometric conservation law (GCL) shall be satisfied as a precondition for this transformation, otherwise some negative effects (such as violation of free-stream conservation, numerical oscillation) may appear [5–19]. Recently, lots of work has been devoted to making it possible for high-order FDSs to solve problems with complex geometries. In order to ensure the GCL for high-order FDSs, CMM which requires $\delta^1=\delta^2$ has been proposed in [20] and $\delta^3=\delta^2$ is also suggested with respect to numerical errors. It is proved that CMM has an effect of improving spatial accuracy for FDSs [21]. In fact, the metrics have many equivalent conservative forms. Compared with the face vectors and control volumes of finite volume schemes (FVSs), Deng et al. [22] proposed SCMM, which uses symmetrical conservative metrics, symmetrical conservative Jacobian and $\delta^1=\delta^2=\delta^3$ and showed the geometric meaning of the discrete metrics and Jacobians based on SCMM for second-order finite difference operators.

Deng et al. have applied SCMM for weighted compact nonlinear scheme (WCNS) [23] in [22] and a new scheme named hybrid cell-edge and cell-node dissipative compact scheme (HDCS) was derived in [24]. They have also shown that high-order FDSs based on SCMM are robust for nonsmooth grids and have good physical properties for solving problems with complex geometries [24, 25]. SCMM may not be equally effective with WENO scheme which takes nonlinear interpolation of fluxes. Nonomura et al. demonstrated that the standard finite difference WENO scheme could not satisfy the GCL [26]. To make the WENO scheme satisfy the GCL, some modifications can be taken [17, 27], where the idea of cancellation of error is essentially the same as that in WCNS and the result schemes are not the standard WENO schemes. Nonomura et al. devided the standard WENO scheme into the consistent central differencing part and the numerical dissipation part to make the WENO scheme to preserve the freestream and some comparisons between the WENO scheme and WCNS were also conducted [28]. While for high-order flux-reconstruction/correction procedure via reconstruction (FR/CPR) schemes, SCMM identity can be easily applied to make the schemes to satisfy GCL [29, 30]. Based on SCMM, Abe et al. [31] discussed the geometric interpretation of the metrics and Jacobian discretized by a linear high-order finite difference scheme and showed that only the symmetric conservative forms of the discretized metrics and Jacobian have the appropriate geometric structures. However, geometric meaning with connection to FVSs and its influence on the form of the final FDSs have not been analyzed.

On the other side, FVSs have superiority in preserving conservation law and have little reliance on grid quality. However, high-order multi-dimensional reconstructions are difficult and take large computational costs. It is well known that surface vector