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Spectral Collocation Method for a Class of Integro-Differential Equations with Erdélyi-Kober Fractional Operator

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Abstract. The Erdélyi-Kober fractional operators have found a large number of applications in many disciplines such as porous media and viscoelasticity. The purpose of this paper is to express the fractional integro-differential equations with Erdélyi-Kober derivative in terms of a class of nonlinear weakly singular integral equations of mixed type in order to analyze their numerical solvability. The resulting mixed type Volterra equations will have kernels containing both an end point and diagonal singularity, with solutions that their derivatives typically are unbounded. Applications of such problems are described to reformulate the fractional integro-differential equations with Erdélyi-Kober derivative in terms of a particular class of cordial weakly singular integral equations of mixed type. The existence and uniqueness results of solutions under some verifiable conditions on the kernels and nonlinear functions are discussed. The corresponding nonlinear weakly singular equation can be solved numerically in terms of the implicitly linear collocation method. The error analysis of the method is also discussed and the feasibility of the introduced strategy is illustrated by some numerical experiments. The reformulation proposed here might be used to develop a computational method to solve fractional integro-differential equations.

AMS subject classifications: 26A33, 47G20, 45G05, 65R20, 45D05

Key words: Nonlinear weakly singular Volterra integral equation, fractional integro-differential equation, Erdélyi-Kober fractional operator, implicitly collocation method, error analysis.

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1 Introduction

The main feature of this paper is to derive a relation between a class of fractional integro-differential equations with Erdélyi-Kober derivative and the noncompact cordial Volterra integral equations of mixed type. This enables us to analyze the numerical solvability of

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the nonlinear mixed Volterra integral equation for obtaining the equivalent solutions of the corresponding fractional integro-differential equations.

The theory of fractional integro-differential equations has been used extensively in recent years in the development of numerous mathematical models and events of the real world. Concerning the development of theory and methods as well as applications of fractional operator equations, one can see [2, 11, 18, 19, 23, 36, 50].

It is known that, when the fractional integro-differential equation involves special integral terms, the tools of fractional calculus are often used. Among several definitions of fractional calculus, the Riemann-Liouville and Caputo definitions are the most widely used in the literature (see e.g., [24,28,32,39]). However, we will focus here on one particular form the so called Erdélyi-Kober (E-K) fractional integral and derivative operators, which gives rise to a class of cordial Volterra integral equation of mixed type. The Erdélyi-Kober fractional operators have found a large number of applications in many disciplines such as porous media, viscoelasticity and electrochemistry [1, 19, 24, 30]. For the theory and further applications of the Erdélyi-Kober fractional integrals see e.g., [21, 46, 47], and more recently [5, 23, 24, 28, 35, 38].

In literature, numerous research papers and monographs have appeared devoted to theoretical and numerical study of fractional integro-differential equations. Lubich [27] introduced and analyzed the fractional linear multi-step methods for fractional derivatives and integrals and obtained the optimal order of convergence of the proposed methods in terms of the order of fractional derivatives. An algorithm based on backward difference approach as well as the classical one-step Adams method for the quadrature methods have been investigated by Diethelm [6] and Diethelm et al. in [7]. A hybrid collocation method has been introduced for solving fractional integro-differential equations in [31]. In [49], the piecewise polynomial collocation methods were used for solving the weakly singular fractional integro-differential equations with the Caputo fractional differential operator. The CAS wavelet method was considered for solving a class of fractional integro-differential equation with a weakly singular kernel in [33]. A novel boundary meshless approach was studied by Fu et al. [13], in connection with a family of fractional integro-differential equations in terms of the time fractional diffusion problems. In [14], the method of approximate particular solutions based on an alternative RBF method has been applied for constant and variable order time fractional diffusion models. The Sinc-Galerkin method [37] has developed to approximate solution of fractional Volterra–Fredholm integro-differential equations with weakly singular kernels. More recently, Fu et al. in [15], applied the collocation Trefftz scheme with high-order T-complete functions in order to obtain a semi-analytical solution of multi-term time fractional diffusion-wave equations with boundary-only collocation method in Laplace domain.

Let us consider the following fractional integro-differential equation

$$\begin{cases} (D_\beta^{\gamma,\delta}y)(t)=g(t)+a(t)y(t)+\displaystyle\int_0^t k(t,s)G\big(s,y(s)\big)ds, & t\in I=[0,T], \quad \gamma\in\mathbb{R}, \\ y^{(k)}(0)=c_k, & k=0,1,\cdots,n-1, \end{cases} \tag{1.1}$$