Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **12**, No. 2, pp. 336-361

DOI: 10.4208/aamm.OA-2018-0198
April 2020

An Alternating Direction Method of Multipliers for Optimal Control Problems Constrained with Elliptic Equations

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Received 17 September 2018; Accepted (in revised version) 8 March 2019

Abstract. In this paper, we propose an efficient numerical method for the optimal control problem constrained by elliptic equations. Being approximated by the finite element method (FEM), the continuous optimal control problem is discretized into a finite dimensional optimization problem with separable structures. Furthermore, an alternating direction method of multipliers (ADMM) is applied to solve the discretization problem. The total convergence analysis which includes the discretization error by FEM and iterative error by ADMM is established. Finally, numerical simulations are presented to verify the efficiency of the proposed method.

AMS subject classifications: 49M37, 65K10, 65M60, 90C33

Key words: Optimal control problem, elliptic equation, finite element method, ADMM.

1 Introduction

The optimal control problem driven by partial differential equations (PDEs) played important roles in industrial, medical and economical applications, etc.. Therefore, how to solve this kind of problems has become one of the hottest topics in the field of the optimal control, optimization and numerical solutions of PDEs. Furthermore, there emerged fruitful research results on mathematical structures, optimization algorithms and discretization techniques. The theoretical analysis and numerical methods for the optimal control problems with PDEs have been developed rapidly.

In this paper, we shall deal with the elliptic optimal control problems and focus on the construction of efficient numerical methods for this kind of problems. The general

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form of the concerned problem is given by

$$\text{(P)} \qquad \begin{cases} \min\limits_{(u,p)} J(u,p) = \dfrac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \dfrac{\gamma}{2}\|p-p_0\|^2_{L^2(\mathcal{B})} \\ \quad \text{subject to} \quad e(u,p)=0, \\ \qquad\qquad\qquad p \in P_{ad}, \end{cases} \tag{1.1}$$

where, u and p denote the state variable and the control variable respectively. Ω is a convex bounded polygonal domain in $\mathbb{R}^2$ with a connected boundary $\partial\Omega$, $\mathcal{B}=\Omega$ (or $\partial\Omega$), J is the objective function, u_d and p_0 are two given functions and $\gamma>0$ is a given regularization parameter. The constraint $e:\mathcal{U}\times\mathcal{P}\to\mathcal{Z}$ is governed by an elliptic equation, which is continuously Fréchet differentiable. Here, $\mathcal{U}=H^2(\Omega)$ (or $H^{\frac{3}{2}}(\Omega)$), $\mathcal{P}=L^2(\Omega)$ (or $H^1(\partial\Omega)$, or $L^2(\partial\Omega)$) stand for Sobolev spaces and $\mathcal{Z}$ is the corresponding dual space $H^{-1}(\Omega)$ (or $(H^1)^*(\Omega)$). The set $P_{ad}\subset\mathcal{P}$ is called the control constraint which is a bounded convex set.

Based on the properties and the structures of the PDE-constrained optimization problems such as (1.1), the researchers have drawn two kinds of approaches for solving these problems: optimize-then-discretize and discretize-then-optimize [10]. It is easy to verify that these two strategies are equivalent when the discretised system of the continuous optimality conditions coincides with the optimality conditions for the discretised optimization problem, but they have differences in terms of system structure, which could provide opportunity for investigators to design algorithms along different directions. The algorithms based on optimize-then-discretize strategy mainly include the steepest descent method, Newton's method, sequential quadratic programming (SQP) method etc.. They are mainly developed for unconstrained control problems [2,15,23]. While the projection method, primal dual active set (PDAS) method, semi-smooth Newton (SSN) method etc. are designed for constrained control problems [10, 19, 21]. Among these methods, the steepest descent method, projection method and PDAS method have better performance on global convergence property than Newton-type methods, but they have only convergence rate of order one. On the contrary, although the convergency of Newton-type methods heavily depends on the selection of the initial value, they have local convergence of order two. The algorithms based on discretize-then-optimize strategy mainly include the extragradient method, approximate proximal point algorithm (APPA), projected gradient-based method, dual ascent method and so on [9, 17, 22, 24, 25, 27]. Theoretically, these methods are all very effective, however they would become applicable only under some strict conditions. For the extragradient method, it is indispensable to assume that the objective function is differentiable and the gradient satisfies Lipschitz condition. In order to ensure the convergence, APPA needs to set some criteria for the error sequence generated by the iterative solution and the exact solution. Both the projected gradient-based method and dual ascent method require the objective function to be strictly convex. How to construct an algorithm which has fast convergence as well as global convergence property under more general conditions becomes more and more urgent.