Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **4**, No. 4, pp. 483-495

DOI: 10.4208/aamm.11-m11175
August 2012

Influence of Gravity and Taper on the Vibration of a Standing Column

C. Y. Wang*

Departments of Mathematics and Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA

Received 16 November 2011; Accepted (in revised version) 9 May 2012

Available online 10 July 2012

Abstract. The stability and natural vibration of a standing tapered vertical column under its own weight are studied. Exact stability criteria are found for the pointy column and numerical stability boundaries are determined for the blunt tipped column. For vibrations we use an accurate, efficient initial value numerical method for the first three frequencies. Four kinds of columns with linear taper are considered. Both the taper and the cross section shape of the column have large influences on the vibration frequencies. It is found that gravity decreases the frequency while the degree of taper may increase or decrease frequency. Vibrations may occur in two different planes.

AMS subject classifications: 74K10, 74H45, 74H55

Key words: Vibration, column, taper, weight.

1 Introduction

The standing column under the influence of gravity models towers, tall buildings, free-standing poles and antennas. The stability of a uniform standing column was solved in the nineteenth century by Greenhill [1] using what is now known as Bessel functions. See Wang et al. [2] for a review on column stability. The vibration of a uniform standing column was recently studied by Virgin et al. [3], whose experimental results confirm numerical predictions superbly.

For strength reasons the standing column is usually not uniform but tapered, wide at base and narrow at the top. Dinnik [4] studied analytically the stability of a power-law tapered standing column, whose tip must decrease into a sharp point. For other cases numerical or semi-numerical methods, such as the Ritz method [5,6], finite elements [7], series expansions [8], integral equations [9] must be used.

*Corresponding author.
Email: cywang@math.msu.edu (C. Y. Wang)

There have been many papers on the vibration of a tapered beam without a compressive axial force. See e.g., [10]. However, to the author's knowledge, there are no reports on the important problem of the vibration of a standing tapered column which is affected by gravity. Since no analytic solutions exist when gravity is present, we shall use a highly efficient initial value method adapted from Barasch and Chen [11] and Wang [12].

2 Formulation

The equation for small vibrations of a non-uniform Euler-Bernoulli column subjected to an axial force can be derived by considering an elemental segment or from energy considerations, e.g., [13]

$$\frac{\partial^2}{\partial x'^2}\Big(EI(x')\frac{\partial^2 y'}{\partial x'^2}\Big)+\frac{\partial}{\partial x'}\Big(F(x')\frac{\partial y'}{\partial x'}\Big)+\rho(x')\frac{\partial^2 y'}{\partial t'^2}=0. \tag{2.1}$$

Here (x', y') are the longitudinal and transverse coordinates of the column (origin at the base), EI is the flexural rigidity, F is the axial force, ρ is the mass per length and t' is the time. Now for a free standing column of height L

$$F=g\int_{x'}^{L}\rho(x')dx', \tag{2.2}$$

where g is the gravitational acceleration. Let

$$EI(x')=EI_0 l(x'), \qquad \rho(x')=\rho_0 r(x'), \tag{2.3}$$

where EI_0 is the maximum of EI and ρ_0 is the maximum of ρ, both occurring at the base at $x'=0$. Consider a harmonic vibration with frequency ω'

$$y'=w'(x')e^{i\omega' t'}. \tag{2.4}$$

Normalize all lengths by the column length L, the time by $L^2\sqrt{\rho_0/EI_0}$ and drop primes. Eq. (2.1) becomes

$$\frac{d^2}{dx^2}\Big[l(x)\frac{d^2 w}{dx^2}\Big]+\beta\frac{d}{dx}\Big[\int_x^1 r(x)dx\frac{dw}{dx}\Big]-\omega^2 r(x)w=0. \tag{2.5}$$

Here

$$\beta=\frac{g\rho_0 L^3}{EI_0}, \qquad \omega=\omega' L^2\sqrt{\rho_0/EI_0} \tag{2.6}$$

are non-dimensional parameters representing gravity force and frequency respectively. At the base of the beam, the column is clamped

$$w(0)=0, \qquad \frac{dw}{dx}(0)=0. \tag{2.7}$$