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A New Robust High-Order Weighted Essentially Non-Oscillatory Scheme for Solving Well-Balanced Shallow Water Equations

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Abstract. A new simple and robust type of finite difference well-balanced weighted essentially non-oscillatory (WENO) schemes is designed for solving the one- and two-dimensional shallow water equations with or without source terms on structured meshes in this paper. Compared with the classical WENO schemes [5] in this field, the set of linear weights of these new WENO schemes could be chosen arbitrarily with one constraint that their summation equals one, maintain the optimal order of accuracy in smooth regions and keep essentially non-oscillatory property in non-smooth regions. For the shallow flow problems with smooth or discontinuous bed, we combine with the well-balanced procedure for balancing the flux gradients and the source terms and then these new WENO schemes with any set of linear weights will satisfy the exact C-property for still stationary solutions and maintain the other advantages of other high-order WENO schemes at the same time. Some benchmark numerical examples are performed to obtain high-order accuracy in smooth regions, keep exact C-property, sustain good convergence property for some steady-state problems and show sharp shock transitions by such new type of finite difference WENO schemes.

AMS subject classifications: 65M60, 35L65

Key words: Shallow water equation, high-order WENO scheme, well-balanced procedure, exact *C*-property, convergence property.

1 Introduction

The shallow water equations (also termed as the Saint-Venant systems [1,2]) with a non-flat bottom topography are often used to model flows in rivers and coastal areas. And

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they also have wide applications in ocean and hydraulic engineering. For example, the ocean currents in estuaries, lakes and sloping beaches, tidal waves, bore propagation, flood routing in natural and man-made streams, submersion waves and atmospheric flows, among others, can be reasonably well described by shallow water equations [3]. The shallow water equations are an approximation to the full free-surface problem and which can also be obtained from the depth-averaged compressible Navier-Stokes equations(in which the depth plays the role of density).

In recent decades, the research on numerical methods for simulating the solutions of the shallow water equations has attracted many attentions. Without taking into account the actions of the source terms, the system is equivalent to the isentropic Euler systems and we can directly solve it with any different kinds of numerical methods for such hyperbolic conservation laws. However, the properties of the system change a lot due to the presence of the source terms. These systems admit stationary solutions in which nonzero flux gradients are exactly balanced by the source terms in the steady state. A straightforward treatment of the source terms will fail to preserve this balance. Therefore, many well-balanced numerical schemes have been developed for solving such systems with source terms. The most important property in connection with the numerical schemes should be able to correctly treat the shallow water equations proposed by Bermuddze and Vazquez [4]. They introduced the notions of the exact conservation property for the numerical schemes which could preserve the quiescent flow exactly. This property is necessary for maintaining the above mentioned balance, which means that the scheme is "exact" when applied to the stationary case $h+b=constant$ and $hu=0$ [5]. A high-resolution finite volume Godunov scheme was presented for two-dimensional shallow water equations in [6] and combined with the surface gradient method for the treatment of source terms in [7] and the approximate Riemann solvers were considered for one-dimensional framework in [8] and Liang presented a well-balanced Godunov scheme for simulating frictional shallow flows over complex domains involving wetting and drying [9]. In 1998, LeVeque [10] developed a quasi-steady wave propagation algorithm for the bed slope source terms which balanced the source terms and flux gradients. A second order TVD conservative scheme [11], a high-resolution finite volume MUSCL method [12] and a upwind Q-scheme [13] were applied to solve for the shallow water flows. A well-balanced non-oscillatory, high-resolution shock-capturing central scheme was designed in [14, 15]. For other related works, such as the numerical simulations of the three-dimensional shallow flows [16–20], positivity-preserving limiter for the dry states [21–23] on unstructured or adaptive meshes [18, 22, 24, 25] and other numerical methods [25–27], were also addressed in the literature.

Recently, many high-order finite difference, finite volume and finite element numerical methods have been applied to solve for the hyperbolic conservation laws. Such as finite difference or finite volume essentially non-oscillatory (ENO) and weighted ENO (WENO) schemes and discontinuous Galerkin (DG) finite element methods were introduced to this research field. And these schemes are also extended to solve the shallow water equations based on well-balanced procedure. Aizinger and Dawson described a