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Quadratic Finite Volume Method for a Nonlinear Elliptic Problem

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Abstract. In this article, a quadratic finite volume method is applied to solve the nonlinear elliptic equation. Firstly, we construct a finite volume scheme for this nonlinear equation. Then, under certain assumptions, the boundedness and ellipticity of the corresponding bilinear form are obtained. Moreover, we get the optimal error estimates not only in H^1-norm but also in L^2-norm where the optimal error estimate in L^2-norm depends on the optimal dual partition. In addition, the effect of numerical integration is analyzed. To confirm the theoretical analysis, we solve the nonlinear equation by the Newton iteration method and prove the quadratic rate of convergence. The numerical results show the effectiveness of our method.

AMS subject classifications: 65N08, 65N15

Key words: Nonlinear elliptic problem, quadratic finite volume method, optimal error estimates, orthogonal conditions.

1 Introduction

Nonlinear partial differential equations are of great significance not only in theory but also in practical application. For examples, they can be used to simulate problems in mechanics, control processes and economical systems. Therefore, studying the numerical methods of nonlinear equations is of great importance. Finite element method (FEM) is one of the most prevailing numerical methods in the research community. Many researchers solved the nonlinear elliptic equation with FEM. Douglas and Dupont [3] discussed a Galerkin method for a nonlinear elliptic equation and gave the optimal error

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estimate. On the basis of Douglas's work, Liu et al. [4] gave a finite element approximation of this equation and extended the coefficient matrix to a more general case. Hlavacek et al. [35] provided Galerkin approximations of a quasilinear elliptic problem with nonlinear mixed boundary conditions. Abdulle and Vilmart [24] analyzed the effect of numerical quadrature in finite element method for the nonlinear problem. André and Chipot [25] discussed the uniqueness and nonuniqueness for approximating the nonlinear elliptic equations. In a word, the theoretical analysis of the nonlinear elliptic equation with FEM is abundant.

Mixed finite element (MFE) method is a popular numerical method in recent years. Mixed finite element method is closely related to finite volume method. With the help of certain numerical quadratures, one can obtain finite volume schemes from mixed finite element method. In 1996, Baranger et al. [26] presented the way to produce finite volume schemes from MFE method with the Raviart-Thomas element of lowest order. Wheeler et al. [27] proposed a multiscale mortar multipoint flux MFE method for elliptic equations. Bause et al. [28] compared the MFE method with a multi-point flux approximation control volume method. Arbogast and Yotov [29] analyzed a non-mortar MFE method on a union of sub-domains. Radu et al. [30] used the MFE method to discretize some degenerate parabolic equations.

With the property of local conservation, Finite volume method (FVM) [10–12, 15, 19, 22, 23, 31, 33, 34] has been a popular numerical method in scientific and engineering computing. Some authors have studied the nonlinear diffusion equation with FVM. Li [1] analyzed the nonlinear Dirichlet problem with linear element FVM, and obtained the optimal error estimate in H^1-norm. On this basis, Chatzipantelidis et al. [2] provided a new proof of this equation with linear element FVM, and got the optimal error estimates in both H^1-norm and L^2-norm. Recently, Bi et al. [13] obtained the existence and uniqueness of the linear element finite volume approximation for a monotone nonlinear elliptic equation and proposed a posterior error estimator. Zhang and Zhang [7] provided the gradient superconvergence of the linear finite volume method for a nonlinear elliptic problem. In summary, some valuable works have been done about the nonlinear elliptic equation with linear element FVM. However, from existing literatures about nonlinear diffusion equation, we haven't found any articles by means of quadratic element FVM.

There are some articles about quadratic element FVM for linear elliptic equation. In 1991, Tian and Chen [18] firstly proposed the quadratic element FVM on triangular meshes for linear diffusion equation. Later, Liebau [19] presented another dual partition for the quadratic element FVM and gave the optimal error estimate in H^1-norm. Xu et al. [14] analyzed quadratic finite volume method for a linear elliptic equation and proved the ellipticity of bilinear form under some weak conditions on the grid. Afterwards, Chen et al. [5] gave a unified framework to analyze the uniform ellipticity of bilinear forms for higher-order finite volume schemes on triangular meshes. Zou [23] proposed an unconditionally stable quadratic finite volume scheme for elliptic equations. Vogel et al. [32, 33] presented some finite volume schemes with arbitrary high order. Radu et al. [31] studied contaminant transport in heterogeneous porous media with high-order fi-