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A Bilinear Petrov-Galerkin Finite Element Method for Solving Elliptic Equation with Discontinuous Coefficients

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Abstract. In this paper, a bilinear Petrov-Galerkin finite element method is introduced to solve the variable matrix coefficient elliptic equation with interfaces using non-body-fitted grid. Different cases the interface cut the cell are discussed. The condition number of the large sparse linear system is studied. Numerical results demonstrate that the method is nearly second order accurate in the L^∞ norm and L^2 norm, and is first order accurate in the H^1 norm.

AMS subject classifications: 65N30

Key words: Petrov-Galerkin finite element method, jump condition, bilinear.

1 Introduction

Interface problem has attracted much attention from researchers of various disciplines because it is involved in many research areas, such as environmental science, physics, fluid dynamics and biological mathematics. Elliptic problem with internal interfaces is the basic form of interface problem. The most challenging part for solving interface problem is that its governing equations has discontinuous coefficients at interfaces and sometimes singular source term exists. Standard finite element or finite difference method are not suitable for this situation if non-body-fitted grid is used. Designing highly accurate and efficient methods for these problems are desired. Since the pioneering work of Peskin in 1977 [1], a large number of numerical methods are designed to solve interface problems.

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The immersed boundary method was proposed by Peskin to study blood flow through heart valves [2]. This method uses a numerical approximation of the delta-function, which smears out the solution on a thin finite band around the interface. In [3], it was combined with the level set method, resulting in a first order numerical method that is simple to implement, even in multiple spatial dimensions. In [4–6], efforts were made to achieve higher order accuracy of the immersed boundary method. Second order convergence are obtained by considering the interaction of a viscous incompressible flow and an anisotropic incompressible viscoelastic shell. This method has been extensively used in engineering computations due to its simplicity, efficiency and robustness [7–9].

In [10–14], LeVeque and Li proposed the immersed interface method (IIM) to solve elliptic equations with discontinuous coefficients and singular source term. This method is based on the finite difference method under the Cartesian grid. Standard finite difference or finite element method is employed away from the interface, while the grid points or elements near the interface are amended by the interface condition. This method has been successfully applied to incompressible Stokes equation and Navier-Stokes equations with singular source term.

Wei et al. pays attention to interface problems with geometry singularity and developed the second order accurate method called matched interface and boundary method [15–17]. This method has been successfully applied to biomathematics research on molecular level. Besides, there are other numerical methods for interface problems from the finite difference/volume perspective, including the boundary condition capturing method [18], the embedded boundary method [19], the Cartesian grid method [41], and so on. The Cartesian grid method [41] developed by Johansen and Colella is a second-order finite volume method on Cartesian grids for the variable coefficient Poisson equation on irregular domains. In [40], the finite volume method is used to solve elliptic equations with variable and discontinuous coefficients. With nonhomogeneous jump conditions, the method can deliver a second order accurate result in the L^2 and L^∞ norm.

Researchers also propose some methods from the finite element perspective. In [20], Chen and Zou considered the finite element method with fitted mesh for solving second order elliptic and parabolic interface problems, sub-optimal error estimates can be achieved for smooth interfaces. The immersed finite element method [34–37] is based on uniform triangulations of Cartesian grids, while the local basis functions are constructed according to the interface jump conditions. In [33], the immersed finite element method is developed to solve elliptic interface problems with non-homogeneous jump conditions. The basic idea is to locally add piecewise polynomials that can approximate the non-homogeneous flux jump condition. For the adaptive immersed interface method [22] and the extended finite element method [23–25], the mesh generation does not rely on the interface, while the construction of finite element space relies on the jump condition of the interface. In [26,28], the non-traditional finite element method is proposed to solve interface problems with variable coefficient and sharp-edged interface. In [27,31,32], the method is further analyzed and extended to three dimensions. The non-traditional finite element method is simple and easy to implement, it is extended to solve elasticity inter-