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Balanced Truncation Based on Generalized Multiscale Finite Element Method for the Parameter-Dependent Elliptic Problem

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Abstract. In this paper, we combine the generalized multiscale finite element method (GMsFEM) with the balanced truncation (BT) method to address a parameter-dependent elliptic problem. Basically, in progress of a model reduction we try to obtain accurate solutions with less computational resources. It is realized via a spectral decomposition from the dominant eigenvalues, that is used for an enrichment of multiscale basis functions in the GMsFEM. The multiscale bases computations are localized to specified coarse neighborhoods, and follow an offline-online process in which eigenvalue problems are used to capture the underlying system behaviors. In the BT on reduced scales, we present a local-global strategy where it requires the observability and controllability of solutions to a set of Lyapunov equations. As the Lyapunov equations need expensive computations, the efficiency of our combined approach is shown to be readily flexible with respect to the online space and an reduced dimension. Numerical experiments are provided to validate the robustness of our approach for the parameter-dependent elliptic model.

AMS subject classifications: 35J25, 65N12, 65N30

Key words: Generalized multiscale method, balanced truncation, parameter dependent, eigenvalue decomposition, Lyapunov equation.

1 Introduction

A variety of research has been devoted to the developments of model reduction for high simulation, optimal control, and engineering design. Many scientific applications in-

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clude the flow models in porous media and conduction models in composite materials. These problems are inherently referred to as multiscale in nature, and they exhibit heterogeneous and high contrast behaviors. Traditional methods such as finite difference method, finite element method, or discontinuous Galerkin method, would require the use of very fine meshes to fully resolve the multiscale nature of media. To an end, the practical applications are limited by the computational power. Therefore the interest in developing efficient multiscale and model reduction methods for more computational efficiency and accuracy are of particular interest.

There are many literatures on model reductions, see e.g. [1–3]. Model reduction performs the discretization on a coarse grid, and it may be capable of incorporating fine-scale features into coarse-scale schemes. Multiscale methods are used to solve a variety of computational models. The construction of coarse space is involved in which solutions are sought with the span of multiscale basis functions. Multiscale Finite Element Method (MsFEM, [4]), Heterogeneous Multiscale Method (HMM, [5]) and related approaches [6–10] work for a variety of applications and offer an advantage in parallel computings.

Progress has been made in recent years in developing the multiscale computations. The Generalized Multiscale Finite Element Method (GMsFEM, see [11]) is a generalization of the standard MsFEM, which uses appropriate snapshots and spectral decompositions for the additional enrichment of multiscale basis functions. Its main idea is to systematically enrich the initial coarse space with the eigenvectors of local spectral problems, and gradually accounts for more fine scale details. The enrichment is performed on a spectral decomposition by the fact that its computational efficiency has been validated the online space construction for any input parameter is fast and it can be reused for any force and boundary. Proper Orthogonal Decomposition (POD, see [2, 12–14]), is effective to be used to find a low rank approximation to a Hilbert space, which is spanned by the snapshots. In the case of matrix approximation, POD is basically Singular Value Decomposition (SVD). And POD has been applied widely for a number of linear and nonlinear problems.

The methodology of GMsFEM has been used in many recent studies [15–25]. Chung, Efendiev and Li [15] derived an a-posteriori error indicator to develop an adaptive enrichment for high-contrast flow models. A parameter-dependent, single-phase flow is studied in [16], in which GMsFEM is used as a local model reduction, and BT is used as a global model reduction. Hou and Liu [20] provided the harmonic multiscale basis functions with an optimal approximation, and through singular value decompositions of some oversampling operators a good efficiency is achieved. In [22] the GMsFEM is combined with variable-separation techniques, and it presents an iterative algorithm for solving the parameter independent PDEs repeatedly. Jiang and Li [24] proposed a model sparse representation based on the mixed GMsFEM with elliptic random inputs, which improves the online computation and the problem output. Transport flow problem in perforated domains are considered in [25], a mixed Petrov-Galerkin GMsFEM formulation is used to guarantee mass conservation and stability in model reductions.