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Fourth-Order Compact Split-Step Finite Difference Method for Solving the Two and Three-Dimensional Nonlinear Schrödinger Equations

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Abstract. In this paper we show a fourth-order compact split-step finite difference method to solve the two and three-dimensional nonlinear Schrödinger equations. The conservation properties and stability are analyzed for the proposed scheme. Numerical results show that the method can provide accurate and stable solutions for the nonlinear Schrödinger equation.

AMS subject classifications: 65M15, 65Y20

Key words: Nonlinear Schrödinger equation, operator splitting method, compact split-step finite difference method, conservation law, stability.

1 Introduction

The nonlinear Schrödinger (NLS) equation has been used extensively in underwater acoustics, quantum mechanics, plasma physics, nonlinear optics and electromagnetic wave propagation [1–4]. In this paper, we consider the following NLS equation

$$i\frac{\partial u(\mathbf{x},t)}{\partial t}=-\frac{1}{2}\nabla^2 u+V(\mathbf{x})u+\beta|u|^2u,\quad \mathbf{x}\in\Omega=\mathbf{R}^d(d=2,3),\quad t\geq 0, \tag{1.1}$$

with the initial and boundary conditions

$$u(\mathbf{x},0)=u_0(\mathbf{x}),\qquad \mathbf{x}\in\Omega=\mathbf{R}^d, \tag{1.2a}$$

$$u(\mathbf{x},t)=0,\quad \mathbf{x}\in\partial\Omega,\qquad t\geq 0, \tag{1.2b}$$

where $\partial\Omega$ is the boundary of Ω, $V(\mathbf{x})$ is an arbitrary real-valued potential function, β is a real constant, and $i=\sqrt{-1}$. It is well known that the NLS equation has two standard conserved quantities (mass and energy):

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Proposition 1.1. If the wave function u is the solution of Eq. (1.1) then this wave function satisfies the following conservation laws:

$$\text{Mass conservation}: \; Q(t)=\int_{\mathbf{R}^d}|u(\mathbf{x},t)|^2d\mathbf{x}=\int_{\mathbf{R}^d}|u(\mathbf{x},0)|^2d\mathbf{x}=Q(0), \tag{1.3a}$$

$$\begin{aligned}\text{Energy conservation}: \; E(t)&=\int_{\mathbf{R}^d}\left[\frac{1}{2}|\nabla u(\mathbf{x},t)|^2+V(\mathbf{x})|u(\mathbf{x},t)|^2+\frac{\beta}{2}|u(\mathbf{x},t)|^4\right]d\mathbf{x}\\ &=\int_{\mathbf{R}^d}\left[\frac{1}{2}|\nabla u(\mathbf{x},0)|^2+V(\mathbf{x})|u(\mathbf{x},0)|^2+\frac{\beta}{2}|u(\mathbf{x},0)|^4\right]d\mathbf{x}\\ &=E(0).\end{aligned} \tag{1.3b}$$

The problem now is how to design a stable and efficient numerical method for Eq. (1.1), which also satisfies discrete conservation laws.

There have different kinds of numerical methods to solve the Schrödinger equations, such as the finite difference methods [5–7]. Bao and Cai [8] established uniform error estimates of finite difference methods for the NLS equation perturbed by the wave operator. Chang et al. [9] studied several finite difference schemes and compared these different schemes for the generalized NLS equation. Kurkinaitis and Ivanauskas [10] investigated several types of finite difference schemes for solving a system of the NLS equations. Sulem et al. [11] proposed several finite difference schemes including spectral method to study the singular solutions to the two-dimensional Cubic NLS equations.

Recently, further improvements on a series of high-order compact methods have been achieved. It is shown that the high-order compact difference schemes play an important role in the simulation of high frequency wave phenomena. Because of their high accuracy, compactness and economic resource in scientific computation, the high-order compact methods are popular over the past decades [12–18]. Xie et al. [19] used some high-order compact finite difference schemes for the numerical solution of one-dimensional NLS equations. Gao and Xie [20] proposed a fourth-order alternating direction implicit compact finite difference scheme for two-dimensional Schrödinger equations. Wang et al. [21] proposed a forth-order compact and energy conservative difference scheme for the two-dimensional NLS equation. In fact, for any compact difference schemes of nonlinear equations, especially those in two or three dimension, there are very few results on unconditional convergence. Wang and Guo [22] established the error estimates of two compact difference schemes for one-dimensional NLS equation without any restrictions on the mesh ratios, but the method cannot be extended directly to high dimension case.

In this paper, we will give a compact split-step finite difference (SSFD) method for solving the NLS equation in two and three-dimension. This method first splits the equation into a linear part and a nonlinear part. Then, for the linear part, we solve it using compact finite difference method. For the nonlinear part, we can solve it exactly. We will test the numerical accuracy and show that the method is unconditionally stable and satisfies the discrete conservation laws.

The SSFD method was pointed early by Weideman and Herbst [23]. Adhikari and