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# Asymptotic Analysis of Travelling Wave Solutions in Chemotaxis with Growth

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**Abstract.** Mass migration of cells (via wave motion) plays an important role in many biological processes, particularly chemotaxis. We study the existence of travelling wave solutions for a chemotaxis model on a microscopic scale. The interaction between nutrients and chemoattractants are considered. Unlike previous approaches, we allow for diffusion of substrates, degradation of chemoattractants and cell growth (constant and linear growth rate). We apply asymptotic methods to investigate the behaviour of the solutions when cells are highly sensitive to extracellular signalling. Explicit solutions are demonstrated, and their biological implications are presented. The results presented here extend and generalize known results.

**AMS subject classifications**: 34D20, 37N25, 92B05

**Key words**: Lie symmetries, velocity-jump process, travelling waves, asymptotic methods.

## 1 Introduction

Chemotaxis is the process whereby cells direct their motion in response to extracellular signalling. The earliest recorded observation of chemotaxis of bacteria occurred in the late 1800s [5, 12, 13, 35]. In his experiment, Adler [1–3] observed the formation of travelling bands of bacteria when he injected a population of cells (*E coli*) at one end of a capillary tube containing oxygen and nutrients. Cells consumed nutrients and excreted a gradient of signal; thereafter moving in response to the signal. As the concentration of the oxygen was inadequate to oxidise all the nutrients, two sharp bands of cells, visible to the naked eye, formed. The first band of cells created a gradient in the concentration of oxygen, while the second band did so for the concentration of nutrients. Both bands swum towards higher concentrations. Mathematical models have been developed to describe chemotaxis.

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The continuum Keller-Segel (K-S) model has become the most common way to represent the chemotactic behaviour on a macroscopic (population-based) point of view. Cell proliferation was not included in the K-S model, as it occurs in some cases over a longer timescale than the duration of many *in vivo* experiments [44]. Keller and Segel [21] shown that a singularity in the chemotactic coefficient is necessary in order to produce the band behaviour (travelling wave solutions) under zero cellular growth/death. To check the validity of the K-S model, Scribner et al. [37] performed numerical simulations and compared their results with Adler's (1966) experimental results under different initial conditions. They provided some forms of $\mu(s)$, $\chi(s)$ and $k(s)$, all dependent on a critical attractant concentration level $a$, that produced both uniform and non-uniform bands of bacteria. Many interesting results (both mathematics and applications) on travelling wave solutions in chemotaxis have recently been obtained by Wang [45].

From an individual perspective, Patlak [33] was the first to propose a chemotaxis model. His model portrayed the random walk process of a particle with persistence of direction, and external bias. In the case where the particles alternatively run (to move forward) and tumble (probably to change the direction), the velocity jump process derived from the stochastic process is appropriate to describe the motion [32]. In the case of no interaction between particles, Alt [4] and Othmer et al. [32] derived a model that employs a transport equation for velocity jump processes as follows:

$$\frac{\partial}{\partial t}p(x,v,t)+v\cdot\nabla p(x,v,t)=\lambda\int_V T(v,v')p(x,v',t)dv', \tag{1.1}$$

where $p(x,v,t)$ is the density of particles at position $x\in\Omega\subset\mathbb{R}^N$, moving with velocity $v\in V\subset\mathbb{R}^N$ at time $t\geq 0$, $\lambda$ is the turning rate, and $T(v,v')$ is the turning kernel standing for the probability of a velocity jump from $v'$ to $v$ if a jump occurs. It was assumed in (1.1) that the choice of the new velocity does not depend on the run length. The intracellular dynamic of cells was later considered to study the signal transduction and metabolism effect [43, 47].

Cell growth and death have often been overlooked in many of the mathematical models of chemotaxis, though they play a biologically significant role in the behaviour of systems. In fact, Budrene et Berg [9] observed that cell growth is crucial for the propagation of the swarm ring and the formation of new aggregates (bands of cells). They also observed that *E coli* cells grow at an approximately constant rate over the concentration range of succinate $0.5-7$mM. Elliott et al. [11] engineered a 3D *in vitro* novel tumor model that allowed the proliferation and spreading of *E coli* cells to invade and interact with bacterial-tumor cells. The effects of cell growth on the behaviour of the solutions has also received a mathematical treatment [22, 24–26, 30]. From the population perspective, Kennedy and Aris [22] found a certain growth function that gave birth to travelling wave solutions of constant speed. Unlike the Keller and Segel's [21] results, Lauffenburger et al. [25] included bacterial growth and death, and assumed that bacteria move by diffusion. They obtained travelling wave solutions (in non *in vivo* experiments) irrespective of the chemotactic coefficient. From the cell-based perspective, Franz et al. [16] also consid-