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A High-Efficient Algorithm for Parabolic Problems with Time-Dependent Coefficients

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Abstract. A high-efficient algorithm to solve Crank-Nicolson scheme for variable coefficient parabolic problems is studied in this paper, which consists of the Function Time-Extrapolation Algorithm (FTEA) and Matrix Time-Extrapolation Algorithm (MTEA). First, FTEA takes a linear combination of previous l level solutions ($U^{n,0}=\sum_{i=1}^{l} a_i U^{n-i}$) as good initial value of U^n (see Time-extrapolation algorithm (TEA) for linear parabolic problems, J. Comput. Math., 32(2) (2014), pp. 183–194), so that Conjugate Gradient (CG)-iteration counts decrease to $1/3\sim1/4$ of direct CG. Second, MTEA uses a linear combination of exact matrix values in level L, $L+s$, $L+2s$ to predict matrix values in the following $s-1$ levels, and the coefficients of the linear combination is deduced by the quadric interpolation formula, then fully recalculate the matrix values at time level $L+3s$, and continue like this iteratively. Therefore, the number of computing the full matrix decreases by a factor $1/s$. Last, the MTEA is analyzed in detail and the effectiveness of new method is verified by numerical experiments.

AMS subject classifications: 65M06, 65M12, 65B05, 65N22

Key words: Crank-Nicolson scheme, Time-Extrapolation, CG-iteration, variable coefficient parabolic.

1 Introduction

Consider a variable coefficient parabolic problem with initial boundary value conditions in a cylindric domain $Q=\Omega\times J$:

$$\begin{cases} u_t-\nabla(a(x,t)\nabla u)=f(x,t), & (x,t)\in Q=\Omega\times J, \quad J=(0,T), \\ u(x,t)=0, & x\in\partial\Omega, \quad t\in J=(0,T), \\ u(x,0)=\varphi(x), & x\in\Omega, \end{cases} \tag{1.1}$$

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where $a(x,t)$ depends on x and t, but not u. Finite Element Method (FEM) based on Crank-Nicolson (C-N) scheme is widely used to deal with parabolic problems.

However, to solve large scale problems, it needs a huge amount of calculation. After discretizing the equations by C-N scheme time-stepping and FEM, we have a large-scale linear equations in each time level. Therefore, there are two major challenges in numerical solution:

1. How to solve the large system of linear equations in each time level quickly;
2. Stiffness and load matrix (hereinafter referred to matrix) must be recalculated at each time level. The effective approaches to address these issues are discussed as follows.

First, in order to solve linear equations, many methods such as Successive Over Relaxation (SOR), Symmetric Successive Over-Relaxation (SSOR), Conjugate Gradient (CG)-iteration, Preconditioned CG (PCG)-iteration and so on have been proposed. The convergence speed for these methods is different. A simple and practical iterative method CG is used in this paper. Each iteration only requires the multiplication between matrix and vector. We use the compressed storage for sparse matrix, where the zero elements don't be saved and be computed. Therefore, the calculating time is linearly proportional to the number of unknowns. Suppose the initial U^0 is known, CG has good convergence

$$\|U^n-U\|_A \leq 2\rho^n \|U^0-U\|_A$$

after n-time iterations, where the convergence rate

$$\rho=\frac{\sqrt{cond(A)}-1}{\sqrt{cond(A)}+1}.$$

As the condition number of elliptic problems is $cond(A)=\mathcal{O}(h^{-2})$, and $\rho=1-\mathcal{O}(h)$, CG is almost the best in all iterative methods. There are two advantages of CG to solve parabolic problems [11]: 1). The solution U^{n-1} in the previous level is a good initial value for the current level, the error $\|U^n-U^{n-1}\|=\mathcal{O}(k)$; 2). The condition number of the matrix that needs to be inverted in each time-step when a parabolic problem is discretized by C-N scheme is not large. Therefore, CG is more effective in solving the parabolic problem than the elliptic problem. In fact, for the five-point scheme of parabolic equation, the condition number $cond(A)\approx 4r$, $r=ak/(2h^2)$, which is far less than the condition number of corresponding discrete elliptic problems where $Cond(A)=\mathcal{O}(h^{-2})$. Note that the convergence factor of CG-iteration is $\rho\approx e^{-\sqrt{1/r}}$ to solve parabolic equations. Therefore, CG-iteration has high efficiency. However, when the step-size becomes very small, e.g., $h=k\ll 10^{-3}$, the number of iterations for CG will increase greatly and overall computation would take a huge amount of time.

In this paper, the Time-Extrapolation Algorithm (TEA) [11] is used. It takes a linear combination of previous l level solutions,

$$U^{n,0}=\sum_{i=1}^{l} a_i U^{n-i}$$