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Error Analysis and Adaptive Methods of Least Squares Nonconforming Finite Element for the Transport Equations

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Abstract. In this paper, we consider a least squares nonconforming finite element of low order for solving the transport equations. We give a detailed overview on the stability and the convergence properties of our considered methods in the stability norm. Moreover, we derive residual type a posteriori error estimates for the least squares nonconforming finite element methods under H^{-1}-norm, which can be used as the error indicators to guide the mesh refinement procedure in the adaptive finite element method. The theoretical results are supported by a series of numerical experiments.

AMS subject classifications: 65N30

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1 Introduction

The transport equations are solved in industrial applications in order to determine the power distribution of neutrons or photons in nuclear reactors. In particular, the neutron transport equations have important applications in nuclear medicine and radiation medicine. It is well known that the standard Galerkin approximation to these equations leads to oscillations when non-smooth behavior of the unknown variables are not properly resolved. To stabilize this phenomenon, several well-established techniques have been proposed and analyzed in a conforming setting (e.g., least squares method [15, 16], finite volume method [24], residual free bubbles method [6]) as well as in a discontinuous setting (e.g., the discontinuous Galerkin method analyzed in [2, 19, 22]).

In this paper, we are interested in least squares nonconforming finite elements such as the Crouzeix-Raviart finite element. This finite element possesses various interesting

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features. First, the degrees of freedom are localized at the mesh faces, thereby leading to efficient communication and parallelization. Second, Crouzeix-Raviart finite element method has close links with finite volume box schemes; see, e.g., [11, 12] for Darcy's equations and [14,17,18,20,25] for advection-diffusion equations. This property is useful to reconstruct locally the diffusive flux in problems where conservatively properties are important, e.g., pollutant transport. Finally, keeping the mesh fixed, the Crouzeix-Raviart finite element space has approximately two times less degrees of freedom than the first-order Discontinuous Galerkin finite element space.

The purpose of the present work is to propose and analyze a least squares nonconforming finite element method for transport equations. We give a detailed overview on the stability and the convergence properties of our considered methods in the stability norm. Moreover, we derive residual type a posteriori error estimates for the least squares nonconforming finite element methods in H^{-1}-norm. Our a posteriori error bound may serve as a refinement indicator within an adaptive mesh-refining algorithm.

The remainder of this paper is organized as following. In Section 2 some necessary notations and the least squares nonconforming finite element discretizations of transport equations are introduced. Here, we also describe assumptions on the finite element spaces to be fulfilled. In Section 3 we provide the coerciveness of the bilinear form related to the discretizations and study a priori and a posteriori error analysis of the least squares nonconforming finite element method. In section 4 we provide several numerical experiments which support our theory. Finally, in Section 5 we summarize the work presented in this article and draw some conclusions.

2 Notations and preliminaries

In this paper, we shall use the standard notation for Sobolev spaces $W^{m,p}(\Omega)$ and their associated norms and seminorms in [1]. For $p=2$, We denote $H^m(\Omega)=W^{m,p}(\Omega)$, $\|\cdot\|_{m,\Omega}=\|\cdot\|_{m,2,\Omega}$ and $(\cdot,\cdot)$ for the standard L^2 inner product. We shall use the letter C to denote a positive constant which may stand for different values at its different occurrences and is independent of the mesh parameters.

In the paper, we consider transport problem: find $u\in H(\mathcal{L},\Omega)$ such that

$$\mathcal{L}u\equiv\mathbf{a}\cdot\nabla u+bu=f \qquad \text{in } \Omega, \tag{2.1a}$$

$$u=g \qquad \text{on } \Gamma_-, \tag{2.1b}$$

where $H(\mathcal{L},\Omega)=\{v\in L^2(\Omega):\mathcal{L}v\in L^2(\Omega)\}$ denotes the graph space of the partial differential operator $\mathcal{L}$ in $L^2(\Omega)$. Here $\Omega\subset\mathbb{R}^2$ is a bounded domain, $\Gamma_-=\{x\in\partial\Omega:\mathbf{a}\cdot\mathbf{n}(x)<0\}$ is the inflow boundary and $\mathbf{n}(x)$ is the outward unit normal at the point $x\in\partial\Omega$. Analogously, we define the outflow boundary $\Gamma_+=\{x\in\partial\Omega:\mathbf{a}\cdot\mathbf{n}(x)\geq 0\}$. Without loss generality we shall suppose that $\mathbf{a}\equiv(a_1,a_2)$ is a non-zeros constant vector. If $\mathbf{a}$ is not a constant vector, the following analysis can be extended to the smooth and bounded function vector by using the methods of [3,10,19]. f, g and b are bounded functions. It will be assumed that