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Coefficient Jump-Independent Approximation of the Conforming and Nonconforming Finite Element Solutions

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Abstract. A counterexample is constructed. It confirms that the error of conforming finite element solution is proportional to the coefficient jump, when solving interface elliptic equations. The Scott-Zhang operator is applied to a nonconforming finite element. It is shown that the nonconforming finite element provides the optimal order approximation in interpolation, in L^2-projection, and in solving elliptic differential equation, independent of the coefficient jump in the elliptic differential equation. Numerical tests confirm the theoretical finding.

AMS subject classifications: 65N30, 65D05

Key words: Jump coefficient, finite element, L^2 projection, weighted projection, Scott-Zhang operator.

1 Introduction

When studying the finite element solutions of elliptic boundary value problems with discontinuous coefficients, i.e., interface problems, a useful tool is the weighted L^2 projection operator, cf. [5,17,19]. To be specific, we consider a domain Ω which is subdivided into finitely many, bounded, polygonal subdomains $\{\Omega_i, i=1,\cdots,J\}$, in $d=2$ or 3 space-dimension. On each subdomain Ω_i, we are given a positive constant ω_i, and we have a quasi-uniform triangulation $\mathcal{T}_h(\Omega_i)$ of size h on Ω_i, cf. [7], shown by Fig. 1 as an example. Thus each Ω_i, and Ω, is Lipschitz. We further assume the subdomain grids are matching at the interface so that we can define conforming and nonconforming linear finite

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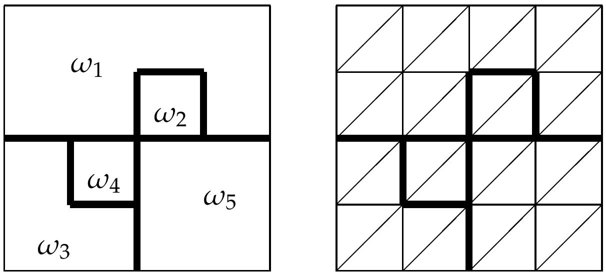

Figure 1: Constant weight each subdomain, and a matching grid.

element spaces on the combined grid $\mathcal{T}_h$ over the domain Ω, [7]:

$$V_h = \{v \in H_0^1(\Omega) \cap C(\Omega) | v|_K \in P_1,\ \forall K \in \mathcal{T}_h\}, \tag{1.1a}$$

$$V_h^{nc} = \{v | v|_K \in P_1,\ \forall K \in \mathcal{T}_h,\ v \text{ is continuous at all mid face } \partial K \text{ and } 0 \text{ at mid face } \partial K \cap \partial \Omega\}. \tag{1.1b}$$

The weighed L^2 and semi-H^1 inner products are defined by

$$(u,v)_{L^2_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i \int_{\Omega_i} uv dx, \tag{1.2a}$$

$$(u,v)_{H^1_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i \int_{\Omega_i} \nabla u \cdot \nabla v dx. \tag{1.2b}$$

The induced norms are denoted by $\|\cdot\|_{L^2_\omega}$ and $|\cdot|_{H^1_\omega}$, respectively. The full H^1 weighted norm is $\|\cdot\|^2_{H^1_\omega} = |\cdot|^2_{H^1_\omega} + \|\cdot\|^2_{L^2_\omega}$. The weighted L^2 projection $Q_h^\omega : L^2(\Omega) \mapsto V_h$ is defined by

$$(Q_h^\omega u, v)_{L^2_\omega(\Omega)} = (u,v)_{L^2_\omega(\Omega)}, \quad \forall v \in V_h. \tag{1.3}$$

The following important theorem is proved by Bramble and Xu in 1991.

Theorem 1.1 (see [5]). *If for all i, the $(d-1)$-dimensional Lebesgue measure of $\partial\Omega_i \cap \partial\Omega$ is positive, then for all $u \in H_0^1(\Omega)$,*

$$\|u - Q_h^\omega u\|_{L^2_\omega(\Omega)} + h|Q_h^\omega u|_{H^1_\omega(\Omega)} \le Ch|\log h|^{1/2}|u|_{H^1_\omega(\Omega)}, \tag{1.4}$$

where C is independent of $\{\omega_i\}$.

Trying to show the necessity that all subdomains have a part of boundary $\partial\Omega$, and of the log term in the bound, several examples are constructed by Xu in [18]. However, these examples are constructed by some limit argument where no specific function u can be used in computation to show the sharpness of (1.4). Thus, some people are still