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A Two-Level Preconditioned Conjugate-Gradient Method in Distorted and Structured Grids

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Abstract. In this paper, we propose a new two-level preconditioned C-G method which uses the quadratic smoothing and the linear correction in distorted but topologically structured grid. The CPU time of this method is less than that of the multigrid preconditioned C-G method (MGCG) using the quadratic element, but their accuracy is almost the same. Numerical experiments and eigenvalue analysis are given and the results show that the proposed two-level preconditioned method is efficient.

AMS subject classifications: 65F08, 65N22, 65N30, 65N50

Key words: Precondition, conjugate gradient, multigrid, finite element.

1 Introduction

The multigrid scheme has been widely used to solve the partial differential equations. In the sixties Fedorenko [1,2] developed the first multigrid scheme for approximating the solution of the Poisson equation in a unit square. Since then, other mathematicians extended his idea to general elliptic boundary value problems with variable coefficients; see, e.g., [3]. However, the full efficiency of the multigrid approach was realized after the works of Brandt [4,5] and Hackbusch [6]. These authors also introduced multigrid methods for nonlinear problems such as the multigrid full approximation storage (FAS) scheme [5,7]. Another achievement in the formulation of multigrid methods was the full multigrid (FMG) scheme [5,7], based on the combination of nested iteration techniques and multigrid methods. Multigrid algorithms are now applied to a wide range of problems, primarily to solve linear and nonlinear boundary value problems. A multigrid preconditioned conjugate gradient (MGCG) method has been put forward by Tatebe in [11], which used the multigrid method as a preconditioner for CG method and has a good convergence rate even for the problems on which the standard multigrid method does not converge efficiently. On the

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other hand, Bank and Douglas [16] treated the conjugate gradient method as a relaxation method of the multigrid method. Braess [12] considered these two combinations and reported the conjugate gradient method with a multigrid preconditioning is effective for elasticity problems. Then Tatebe and Oyanagi considered a parallelization of the MGCG method and proposed an efficient parallel MGCG method on distributed memory machines [15]. A class of usefull of solvers based on the multigrid strategy are algebraic multigrid (AMG) methods [8] that resemble the geometric multigrid process utilizing only information contained in the algebraic system to be solved. It is noted that S. Shu et al proposed an algebraic multigrid method for higher order finite element discretizations [13], who also studied AMG method for finite element systems on criss-cross grids [14].

In this paper, we study an efficient multigrid method which can be used in distorted but topologically structured grid. We utilize iterative grid redistribution method, proposed by Ren and Wang in [10], to generate mesh which concentrates in the region where solution has large variation. A quadratic finite element method and a linear finite element method are both employed to discretize the equation and MGCG method is used to solve the discretized system $Au = b$. The result of using quadratic element is more accurate than that of using linear element, while the former costs more CPU time than the latter. We would like to obtain the accuracy of using quadratic element and cost less CPU time. We improve MGCG method and make it more efficient. Our two-level method in the preconditioning step has several crucial parts: (1) take pre-smoothing steps by Gauss-Seidel iteration in the quadratic finite element space; (2) calculate the residue and restrict it on the linear finite element space; (3) use the V-cycle multigrid scheme to solve $Au = r$ (r is the residue); (4) prolongate the solution from the linear element space to the quadratic element space; (5) take post-smoothing steps by Gauss-Seidel iteration in the quadratic element space. The above is different from current MGCG method and will be shown very useful.

In the following, the multigrid preconditioned conjugate gradient method and our new two-level preconditioned C-G method are described in Sections 2 and 3. The efficiency of the two-level method are verified by numerical examples given in Section 4. In Section 5, eigenvalue analysis is presented, which explains why our two-level method is efficient. When the two-level method is used as a preconditioner of the conjugate gradient method, it becomes quite an effective and desirable preconditioner of the conjugate gradient method.

2 Multigrid preconditioned C-G method

The MGCG method is a PCG method that uses the multigrid method as a preconditioner. When a target linear equation is

$$L_l \mathbf{x} = \mathbf{f},$$