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Some Weighted Averaging Methods for Gradient Recovery

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Abstract. We propose some new weighted averaging methods for gradient recovery, and present analytical and numerical investigation on the performance of these weighted averaging methods. It is shown analytically that the harmonic averaging yields a superconvergent gradient for any mesh in one-dimension and the rectangular mesh in two-dimension. Numerical results indicate that these new weighted averaging methods are better recovered gradient approaches than the simple averaging and geometry averaging methods under triangular mesh.

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Key words: Finite element method, weighted averaging, gradient recovery.

1 Introduction

Finite element recovery techniques are post-processing methods that reconstruct numerical approximations from finite element solutions to obtain improved solutions. A classical recovery technique is a simple averaging technique which is as old as the finite element method itself. Whereafter, different kinds of post-processing techniques are developed based on weighted averaging [4–7, 11], local or global projections [3, 8, 13, 17], post-processing interpolation [14, 22], smoothing techniques [9, 10, 18, 20], and the local least-squares methods including the superconvergent patch recovery (SPR) [25–27], the polynomial preserving recovery (PPR) [16, 23] and the superconvergent cluster recovery (SCR) [12].

For the Lagrange element, the gradient of the finite element approximation provides a discontinuous approximation. Two classical techniques of simple averaging

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and geometry averaging are deveopled by engineers to improve the precision of finite element solution. Those well-known recovery techniques construct an gradient approximation for each node by averaging the contributions of the surrounding nodes. These values may be interpolated to obtain a continuous approximation over the whole domain.

In this paper, we propose some new weighted averaging techniques which are named harmonic averaging, angle averaging, and distant averaging for gradient recovery. Together with simple averaging and geometry averaging, we investigate the relationship of all the five weighted averaging methods. We will show analytically that, for any mesh in one-dimension and rectangular mesh in two-dimension, harmonic averaging is a superconvergent gradient recovery method. Under the triangular mesh, we also provide numerical evidence to show that harmonic averaging, angle averaging and distant averaging are performance better than simple averaging and geometry averaging.

The rest of the paper is organized as follows: in Section 2 we describe the construction of these weighted averaging techniques in detail. We analyze harmonic averaging, simple averaging and geometry averaging for one-dimension problems in Section 3, and in Section 4, we investigate harmonic averaging, simple averaging and geometry averaging for two-dimension problems under rectangular mesh. In Section 5, we consider the weighted averaging methods under triangular mesh. Numerical tests illustrating the performance of our new weighted averaging methods are presented in Section 6. Finally, in Section 7, some conclusions and future work are presented.

2 Weighted averaging gradient recovery operators

In this section, we give the definitions of all the five weighted averaging gradient recovery operators including two known called simple averaging and geometry averaging and three new called harmonic averaging, angle averaging and distant averaging. In detail, the simple averaging, geometry averaging and harmonic averaging are defined for both one-dimension problems and two-dimension problems, and angle averaging and distant averaging are only defined for triangular element.

Let $\mathcal{T}_h$ be partition of $\Omega \subset \mathbb{R}^d$ with $d = 1,2$ and S_h be a $\mathcal{C}^0$ finite element space over $\mathcal{T}_h$, and $S_h^d = \prod_{i=1}^d S_h$. Given a finite element function $v \in S_h$, v is piecewise continuous. We firstly define $R_h v$ at each node, where operator $R_h : S_h \to S_h^d$. After defining values of $R_h v$ at all nodes, we obtain $R_h v \in S_h^d$ on the whole domain by interpolation using the original nodal shape functions of S_h.

Firstly, we give the definition of simple averaging, geometry averaging and harmonic averaging in 1D. We consider the problem on the unit interval $I = (0,1)$. Any other interval can be mapped to unit interval by a linear transformation. A subdivision of domain I

$$0 = x_0 < x_1 < x_2 < \cdots < x_N = 1,$$