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Convergence Analysis of Legendre-Collocation Methods for Nonlinear Volterra Type Integro Equations

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Abstract. A Legendre-collocation method is proposed to solve the nonlinear Volterra integral equations of the second kind. We provide a rigorous error analysis for the proposed method, which indicate that the numerical errors in L^2-norm and L^∞-norm will decay exponentially provided that the kernel function is sufficiently smooth. Numerical results are presented, which confirm the theoretical prediction of the exponential rate of convergence.

AMS subject classifications: 65R20, 45J05, 65N12

Key words: Spectral method, nonlinear, Volterra integral equations.

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1 Introduction

The integro-differential equations (IDEs) arise from the mathematical modeling of many scientific phenomena. Nonlinear phenomena, that appear in many applications in scientific fields can be modeled by nonlinear integro-differential equations. This paper is concerned with the nonlinear Volterra integral equations (VIEs) of the second kind

$$y(t)=\int_0^t \widehat{K}(t,\tau,y(\tau))d\tau+\widehat{g}(t), \quad t\in[0,T], \tag{1.1}$$

where kernel function $\widehat{K}:S\times\mathbb{R}\to\mathbb{R}$ (where $S:=\{(t,\tau):0\le\tau\le t\le T\}$) and $\widehat{g}(t):[0,T]\to\mathbb{R}$ are known, function $y(t)$ is the unknown function to be determined. It will always

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be assumed that problem (1.1) possesses a unique solution, $\widehat{K}$ is continuous for all S and Lipschitz continuous with its third argument. We will consider the case that the solutions are sufficiently smooth. Consequently it is natural to implement very high-order numerical methods such as spectral methods for the solutions.

In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations and integro-differential equations, such as the linearization method [1], differential transform method [2], RF-pair method [4], product integration method [30], Hermite-type collocation method [5], semi-analytical-numerical techniques, such as the Adomian decomposition method [36], Taylor expansion approach [3], collocation methods [9] and references therein. Nevertheless, few works touched the spectral approximations to integral-differential equations. Then Chebyshev spectral methods were investigated in [19] for the first kind Fredholm integral equations under multiple-precision arithmetic. However, no theoretical results were provided to justify the high accuracy numerically obtained. Recently, Tang and Xu [32] developed a novel spectral Legendre-collocation method to solve linear Volterra integral equations of (1.1). Xie, Li and Tang [38] developed a spectral Petrov-Galerkin methods for linear Volterra type integral equations. Chen, Li and Tang [16–18] proposed a spectral Jacobi-collocation approximation for linear Volterra integral equations with weakly singular kernels. For nonlinear case, polynomial spline collocation methods for the nonlinear Basset equation is discussed in [12]. Legendre spectral Galerkin method has been proposed to nonlinear Volterra integral equations in [35]. In this paper, the main purpose of this work is to provide Legendre-collocation methods for nonlinear Volterra integral equations and provide a rigorous error analysis which theoretically justifies the spectral rate of convergence. The linear kind of (1.1) was provided in [32], but they pointed out that the rate of convergence seems not optimal, in this paper, the optimal order of convergence $\mathcal{O}(N^{-m})$ is obtained.

The paper is organized as follows. In Section 2, we outline the spectral approaches for (1.1). Some lemmas useful for establishing the convergence results will be provided in Section 3. The convergence analysis will be carried out in Section 4, and Section 5 contains numerical results, which will be used to verify the theoretical results obtained in Section 4. Finally, in Section 6, we end with conclusions and future work.

2 Legendre-collocation method

For a given $N \geq 0$, we denote by $\{\theta_k\}_{k=0}^N$ the Legendre points, and by $\{\omega_k\}_{k=0}^N$ the corresponding Legendre weights. Then, the Legendre-Gauss integration formula is

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=0}^{N} f(\theta_k)\omega_k. \tag{2.1}$$

For the sake of applying the theory of orthogonal polynomials, we use the change of