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A Modified Kernel Method for Solving Cauchy Problem of Two-Dimensional Heat Conduction Equation

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Abstract. In this paper, a Cauchy problem of two-dimensional heat conduction equation is investigated. This is a severely ill-posed problem. Based on the solution of Cauchy problem of two-dimensional heat conduction equation, we propose to solve this problem by modifying the kernel, which generates a well-posed problem. Error estimates between the exact solution and the regularized solution are given. We provide a numerical experiment to illustrate the main results.

AMS subject classifications: 47A52, 65J22

Key words: Ill-posed problem, Cauchy problem, modified kernel method.

1 Introduction

In many industrial applications one wants to determine the temperature or heat flux on the surface of a body, where the surface itself is inaccessible for measurements [1]. The Cauchy problem of the heat conduction equation can be considered as a data completion problem that means to achieve the remaining part information from boundary conditions for both the solution and its normal derivative of the boundary. This sort of problem many occur in a large field of practical applications. In a one-dimensional setting this situation can be modelled as the following problem for the heat equation. Determining the temperature $u(x,t)$ for $0<x\leq 1$ from temperature measurements $u(0,t)=g(t)$ and heat flux measurements $u_x(0,t)=0$, when $u(x,t)$ satisfies

$$\begin{cases} u_t(x,t)=u_{xx}(x,t), & t\geq 0, \quad 0<x<1, \\ u(0,t)=g(t), & t\geq 0, \\ u_x(0,t)=0, & t\geq 0, \\ u(x,0)=0, & 0<x<1. \end{cases} \tag{1.1}$$

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Problem (1.1) has been studied by several authors, see for instance [3, 10] and also [2].

In this paper, motivated by (1.1), we want to extend problem (1.1) to a Cauchy problem of two-dimensional heat conduction equation in a semi-infinite slab, i.e.,

$$\begin{cases} u_t(x,y,t)=u_{xx}(x,y,t)+u_{yy}(x,y,t), & 0<x<1, \quad y>0, \quad t>0, \\ u(0,y,t)=g(y,t), & y\geq 0, \quad t\geq 0, \\ u_x(0,y,t)=0, & y\geq 0, \quad t\geq 0, \\ u(x,y,0)=0, & 0\leq x\leq 1, \quad y\geq 0, \\ u(x,0,t)=0, & 0\leq x\leq 1, \quad t\geq 0, \\ u(x,y,t)|_{y\to\infty} \text{ bounded}, & 0\leq x\leq 1, \quad t\geq 0. \end{cases} \tag{1.2}$$

Due to the complexity of this problem, it is much more difficult to solve Cauchy problem of heat conduction equation in the 2D case. To the knowledge of the authors, there are still very few results on Cauchy problem of 2D heat conduction problem, e.g., the articles by Li and Wang [9], Qian and Fu [12].

In order to apply the Fourier transform, we extend the functions $u(x,\cdot,\cdot)$ to be whole real (y,t) plane by defining them to be zero everywhere in $\{(y,t),\ y<0,\ t<0\}$. We also assume that these functions are in $L^2(R^2)$. Practically, the input data $g(y,t)$ is measured, there will be measured data function $g^\delta(y,t)\in L^2(R^2)$ with measured error which satisfy

$$\|g^\delta - g\|_{L^2(R^2)} \leq \delta, \tag{1.3}$$

the constant $\delta>0$ represents a bound on the measurement error. Our aim is to seek the solution $u(x,y,t)$ form the Cauchy data $[u,u_x]$, given on the line $x=0$.

It is well-known that Cauchy problem is generally ill-posed, i.e., the existence, uniqueness and stability of their solutions are not always guaranteed, see e.g., Hadamard [8]. A small perturbation in the data $g^\delta(y,t)$ may cause a dramatically large error in the corresponding solution $u(x,y,t)$ for $0<x\leq 1$. Such ill-posedness is caused by the perturbation of high frequencies. Thus, an appropriate regularization method is required.

The paper is organized as follows: in Section 2, we demonstrate ill-posedness of a Cauchy problem of 2D heat conduction equation. In Section 3, we propose a modifying kernel method to solve this ill-posed problem and give error estimates between the regularization solution and the exact solution under a priori choice of the regularization parameter. In Section 4, A numerical experiment is given to illustrate the accuracy and efficiency of our method. Finally, we conclude this paper in Section 5.

2 Ill-posedness of a Cauchy problem of 2D heat conduction equation

Here, and in the following sections, $\|\cdot\|$ denotes the $L^2(R^2)$-norm as

$$\|f\| = \Big(\int_{R^2} |f(y,t)|^2 dydt \Big)^{\frac{1}{2}}. \tag{2.1}$$