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A Finite Volume Method Based on the Constrained Nonconforming Rotated Q_1-Constant Element for the Stokes Problem

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Abstract. We construct a finite volume element method based on the constrained nonconforming rotated Q_1-constant element ($CNRQ_1$-P_0) for the Stokes problem. Two meshes are needed, which are the primal mesh and the dual mesh. We approximate the velocity by $CNRQ_1$ elements and the pressure by piecewise constants. The errors for the velocity in the H^1 norm and for the pressure in the L^2 norm are $\mathcal{O}(h)$ and the error for the velocity in the L^2 norm is $\mathcal{O}(h^2)$. Numerical experiments are presented to support our theoretical results.

AMS subject classifications: 65N08, 65N15, 65N30, 76D05

Key words: Stokes problem, finite volume method, constrained nonconforming rotated Q_1 element.

1 Introduction

Let Ω be a bounded, convex and open polygon of $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the following Stokes equations with the homogeneous Dirichlet boundary condition

$$-\Delta\mathbf{u}+\nabla p=\mathbf{f}, \qquad \text{in } \Omega, \tag{1.1a}$$
$$\text{div}\mathbf{u}=0, \qquad \text{in } \Omega, \tag{1.1b}$$
$$\mathbf{u}=0, \qquad \text{on } \partial\Omega, \tag{1.1c}$$

where $\mathbf{u}=(u^1,u^2)$ represents the velocity vector, p is the pressure and $\mathbf{f}$ indicates a prescribed body force. Let $L_0^2(\Omega)$ be the set of all $L^2(\Omega)$ functions over Ω with zero integral mean and let

$$H_0^1(\Omega):=\big\{u\in H^1(\Omega): u=0 \text{ on } \partial\Omega\big\}.$$

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The variational formulation of (1.1a)-(1.1c) is: find a pair $(\mathbf{u},p)\in H_0^1(\Omega)^2\times L_0^2(\Omega)$ such that (see [10])

$$a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)=(\mathbf{f},\mathbf{v}),\qquad \forall\mathbf{v}\in H_0^1(\Omega)^2, \tag{1.2a}$$

$$b(\mathbf{u},q)=0,\qquad \forall q\in L_0^2(\Omega), \tag{1.2b}$$

where

$$a(\mathbf{u},\mathbf{v})=(\nabla\mathbf{u},\nabla\mathbf{v}),\qquad b(\mathbf{v},p)=-(p,\mathrm{div}\mathbf{v}).$$

Finite volume method (FVM) is an important numerical discretization technique for solving partial differential equations, especially for those arising from physical conservation laws including mass, momentum and energy. In general, FVM has both simplicity in implementation and local conservation property, so it has enjoyed great popularity in many fields, such as computational fluid dynamics, computational aerodynamics, petroleum engineering and so on. About some recent development of FVM, readers can refer to the monographs [6,7,9,13,16,19,20,27,28] for details.

In recent years, there have been a lot of studies on the mixed finite element methods and mixed finite volume element methods for the Stokes problem. Among these studies, some lower order quadrilateral finite elements seem to be more attractive, e.g., the conforming bilinear Q_1-P_0 element [7,17,24,26], the conforming bilinear Q_1-Q_1 element [1,14] and nonconforming rotated Q_1-P_0 element [25] with some variants [4,13]. All the approximation finite elements for the velocity need at least four degrees of freedom on each quadrilateral. In [23], Park and Sheen have introduced a P_1-nonconforming quadrilateral element which has only three degrees of freedom on each quadrilateral. Later, Man and Shi [20] proposed the P_1-nonconforming quadrilateral FVM for the elliptic problem by using a dual partition of overlapping type. Following the line of the finite element in [23], Hu and Shi [12] presented a constrained nonconforming rotated Q_1 ($CNRQ_1$) element and applied it to the second order elliptic problem. In [12], the authors also point out that the $CNRQ_1$ element and the P_1-nonconforming element are equivalent on a rectangle since the constraint and the continuity are the same, however, the two elements are different on a general quadrilateral. Afterwards, Hu, Man and Shi [11] and Mao and Chen [21] investigated and analyzed the $CNRQ_1$-P_0 finite element method for the Stokes problem on rectangular meshes. The application of the $CNRQ_1$ element to the nearly incompressible planar elasticity problem can be referred to [11,22]. Meanwhile, Mao and Chen [21] and Liu and Yan [18] discussed the supconvergence of the finite element scheme for the Stokes problem on rectangular meshes.

The purpose of this paper is to investigate a new mixed FVM for solving the Stokes problem on quadrilateral meshes. We will approximate the velocity by the $CNRQ_1$ element (see [12]) based on the primal quadrilateral partition, while the test function space of the velocity is the piecewise constant space associated to the nonoverlapping dual partition. Following the ideas presented in [11,18,21,24], we employ a piecewise constant approximation for the pressure. We also analyze the stability of this