Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **10**, No. 1, pp. 114-137

DOI: 10.4208/aamm.OA-2016-0196
February 2018

A Maximum-Principle-Preserving Third Order Finite Volume SWENO Scheme on Unstructured Triangular Meshes

Yunrui Guo, Lingyan Tang*, Hong Zhang and Songhe Song

Department of Mathematics and Systems Science, and State Key Laboratory of High Performance Computing, National University of Defense Technology, Changsha, Hunan 410073, China

Received 24 November 2016, Accepted (in revised version) 12 May 2017

Abstract. We modify the construction of the third order finite volume WENO scheme on triangular meshes and present a simplified WENO (SWENO) scheme. The novelty of the SWENO scheme is the less complexity and lower computational cost when deciding the smoothest stencil through a simple mechanism. The LU decomposition with iterative refinement is adopted to implement ill-conditioned interpolation matrices and improves the stability of the SWENO scheme efficiently. Besides, a scaling technique is used to circument the growth of condition numbers as mesh refined. However, weak oscillations still appear when the SWENO scheme deals with complex low density equations. In order to guarantee the maximum-principle-preserving (MPP) property, we apply a scaling limiter to the reconstruction polynomial without the loss of accuracy. A novel procedure is designed to prove this property theoretically. Finally, numerical examples for one- and two-dimensional problems are presented to verify the good performance, maximum principle preserving, essentially non-oscillation and high resolution of the proposed scheme.

AMS subject classifications: 65M10, 78A48

Key words: Triangular meshes, WENO, scaling limiter, maximum-principle-preserving.

1 Introduction

We consider the two-dimensional scalar conservation law

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0 \tag{1.1}$$

*Corresponding author.
Email: yunruiguo@nudt.edu.cn (Y. R. Guo)

subject to the initial condition $u(x,y,0)=u_0(x,y)$. WENO schemes are high order numerical methods for solving PDEs which may contain discontinuities, sharp gradient regions and other complex solution structures. The main idea of WENO schemes is to form a weighted combination of several local reconstructions based on different stencils and use it as the final WENO reconstruction. Thus WENO schemes have the ability to achieve high order accuracy in smooth regions while maintaining sharp and essentially monotone shock transitions.

Abgrall first recalled and improved the results of an earlier literature about non-oscillatory reconstruction on unstructured meshes in literature [1]. Later Hu and Shu [2] constructed high order weighted essentially non-oscillatory schemes on unstructured triangular meshes in the finite volume formulation. They presented third order schemes using a combination of linear polynomials and fourth order schemes using a combination of quadratic polynomials. However, oscillations would arise when strong discontinuities are included in the solution. In 2005, Haslbacher proposed a weighted essentially non-oscillatory (WENO) reconstruction algorithm. The stencils are fixed to reduce a high computational cost typically associated with the WENO scheme [3]. For accurate simulations of high Mach number aerodynamic flows with strong discontinuities, Wolf and Azevedo described the implementation and analysis of the high order ENO and WENO schemes applied to high speed flows on unstructured grids in [4]. The advantage of the method is that the reconstruction stencils can be composed by control volumes with any number of edges, e.g., triangles, quadrilaterals and hybrid meshes. Singular unstructured meshes always lead to ill-conditioned coefficient matrices. Therefore, a robust WENO reconstruction procedure was designed to deal with distorted local mesh geometries or degenerate cases when the mesh quality varies for complex domain geometry [5]. Besides, a new family of ENO schemes were derived via Taylor series expansion and solved using a weighted least squares formulation [6], which did not require constructing sub-stencils. Thus it provides a more flexible framework and less sensitivity to the mesh.

There are two types of finite volume WENO schemes on unstructured meshes in the literature above mentioned. The first type consists of WENO schemes whose orders are not higher than that of the reconstruction on each small stencil, such as [6,7]. For this type, the non-linear weights do not contribute to the increase of the order but to the nonlinear stability purely, or to avoid spurious oscillations. The second type consists of WENO schemes whose order is higher than that of the reconstruction on each small stencil, such as [2, 8]. They are more complex than the former that only need to choose the linear weights as arbitrary positive numbers for the better linear stability. In this paper, we focus on the finite volume WENO scheme on unstructured triangular meshes [9] belonging to the first type. The simple strategy of the stencil construction can achieve the third order accuracy using less cells but higher resolution than other WENO schemes on triangular meshes [6,11]. A crucial step in building this scheme is to compute interpolation matrices. However, traditional solvers for linear systems are prone to collapse for ill-conditioned matrices, such as the LU decomposition, the Jacobi iterative method and the Gauss–Seidel