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Stabilized Finite Element Methods for Biot's Consolidation Problems Using Equal Order Elements

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Abstract. Using the standard mixed Galerkin methods with equal order elements to solve Biot's consolidation problems, the pressure close to the initial time produces large non-physical oscillations. In this paper, we propose a class of fully discrete stabilized methods using equal order elements to reduce the effects of non-physical oscillations. Optimal error estimates for the approximation of displacements and pressure at every time level are obtained, which are valid even close to the initial time. Numerical experiments illustrate and confirm our theoretical analysis.

AMS subject classifications: 65M60

Key words: Biot's problem, LBB condition, stabilized method, error estimates, numerical experiments, Terzaghi problem.

1 Introduction

The Biot's consolidation model describes the time-dependent interaction between the deformation of an elastic porous material and the fluid flow inside of it. This problem was first proposed by Terzaghi [1], and summarized by Biot [2–4]. Biot's model is widely used in geomechanics, hydrogeology, petrol engineering and biomechanics. This paper focuses on the quasi-static Biot's consolidation model.

Variational principles for Biot's consolidation problem and finite element approximations based on the Galerkin method were presented in [5–9]. Asymptotic behavior of semi-discrete finite element approximations of the Biot's consolidation problem was discussed in [8]. The authors analyzed the standard mixed Galerkin methods for the Biot's consolidation problem in [9]. Long-time stability was proved since they obtained

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an exponential decay of the error in the initial data in time. Error estimates on the LBB (Ladyzhenskaya-Babuška-Brezzi) stable and LBB unstable spaces combinations were presented. For the LBB stable cases, the error estimates were optimal even close to the initial time. On the other hand, for the unstable cases, especially those with equal-order of interpolation, the lack of stability close to the initial time results from an unstable approximation of the initial condition. Therefore, oscillations of the pressure close to the initial time may happen. Some researches [9, 10] are devoted to working on this issue. Murad and Loula [9] proposed a post-processing technique, which has to use the LBB stable space combinations in the post-processing. Paper [10] presented a penalty stabilized scheme using equal-order linear space, the numerical experiments showed the good stability and convergence of their method, but no further theoretical analysis is given. A fully discrete stabilized discontinuous Galerkin method was proposed in [11], error estimates for the pressure close to the initial time and numerical experiments were not given.

Motivated by the stabilized methods for the Stokes problem [13–19], in this paper we proposed a large class of fully discrete stabilized methods including the method in [10]. By adding a weak consistent term with time derivate of pressure, we obtain additional control of the pressure. Then we establish the error estimates for the velocities and pressure with arbitrary combination of interpolations. The error estimates are optimal even close to the initial time. Numerical experiments illustrate and confirm our theoretical analysis.

An outline of the paper is as follows. In Section 2, we introduce the quasi-static Biot's consolidation model. In Section 3, we propose and analyze the stability of our methods. In Section 4 we give error estimates for our scheme. In Section 5, we give some numerical experiments. In Section 6, we conclude the whole paper.

Throughout this paper, we use C to denote a positive constant independent of Δt and h, not necessarily the same at each occurrence.

2 The quasi-static Biot's consolidation model

Let $\Omega \in \mathbb{R}^d$ $(d=1,2,3)$ be a bounded domain with polygonal or polyhedral boundary $\Gamma=\partial\Omega$. We use $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ to denote the m-order Sobolev spaces on Ω, and use $\|\cdot\|_{m,p}$, $|\cdot|_{m,p}$ to denote the norm and semi-norm on these spaces, respectively. When p=2, we set $\mathbf{H}^m(\Omega)=W^{m,p}(\Omega)$, $\mathbf{H}_0^m(\Omega)=W_0^{m,p}(\Omega)$ and $\|\cdot\|_m=\|\cdot\|_{m,p}$, $|\cdot|_m=|\cdot|_{m,p}$. Denote the inner product of $\mathbf{H}^m(\Omega)$ by $(\cdot,\cdot)_m$ and $(\cdot,\cdot)=(\cdot,\cdot)_0$. Let X denote a Banach space with the norm $\|\cdot\|_X$. We define

$$L^\infty(0,T;X)=\left\{v\in X: ess\ \sup_{0\le t\le T}\|v\|_X^2<\infty\right\}, \tag{2.1a}$$

$$L^2(0,T;X)=\left\{v\in X: \int_0^T \|v\|_X^2 dt<\infty\right\}, \tag{2.1b}$$