Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **7**, No. 6, pp. 818-826

DOI: 10.4208/aamm.2014.m617
December 2015

Transient Diffusion in Triangular Cylinders

C. Y. Wang*

Departments of Mathematics and Mechanical Engineering, Michigan State University, East Lansing, MI 48824, USA

Received 13 May 2014; Accepted (in revised version) 12 November 2014

Abstract. A heated triangular cylinder is suddenly cooled in a constant temperature bath. The transient heat conduction problem is transformed to the Helmholtz equation related to the vibration of membranes. Using the membrane analogy, exact analytic solutions for the transient heat conduction problem for three triangular cross sections are found.

AMS subject classifications: 80A20

Key words: Diffusion, heat conduction, transient, triangular.

1 Introduction

Heat or mass diffusion is important in many diverse applications. One basic problem is the cooling of a heated solid which is suddenly introduced into a bath of lower temperature (quenching). The seminal work of Carslaw and Jaeger [1] discussed analytic solutions for the cooling of cylindrical solids whose cross sections are rectangular, circular, annular, or sectorial. Note that these geometries have boundaries that can be described by separable coordinates, such that decoupled ordinary differential equations result. For other cross sections numerical integrations are usually needed.

In this note we shall study the cooling of three special triangular cylinders. Since triangular boundaries cannot be described by separable coordinates, analytic methods such as separation of variables or Laplace transform described in [1] are difficult to apply. We shall show the cooling of these triangular cross sections is related to the eigenfunctions for the Helmholtz equation. Thus an analogy exists.

*Corresponding author.
Email: cywang@mth.msu.edu (C. Y. Wang)

2 Formulation

The transient heat transfer in a solid cylinder is governed by the diffusion equation

$$\nabla^2 T' = k\frac{\partial T'}{\partial t'}. \tag{2.1}$$

Here, ∇^2 is the Laplace operator in two dimensions, T' is the temperature, k is the thermal diffusivity, and t' is the time. Let the cross section have a (convenient) characteristic length L, original temperature ${T_0}'$ and ambient temperature ${T_a}'$. Normalize all lengths by L, the time and the temperature by

$$t = \frac{t'}{kL^2}, \quad T = \frac{T' - {T_a}'}{{T_0}' - {T_a}'}. \tag{2.2}$$

Eq. (2.1) becomes

$$\nabla^2 T = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) T = \frac{\partial T}{\partial t}, \tag{2.3}$$

where x, y are Cartesian coordinates describing the cross section. The boundary condition is

$$T = 0 \quad \text{on} \ \ S, \tag{2.4}$$

where S is the boundary of the cross section, and initially inside the solid

$$T = 1 \quad \text{at} \ \ t = 0. \tag{2.5}$$

Using separation of space and time variables on Eq. (2.3), the solution can be expressed in series form

$$T = \sum_i A_i \varphi_i(x,y) e^{-\lambda_i^2 t}, \tag{2.6}$$

where A_i are constant coefficients to be determined. Eq. (2.3) then yields the Helmholtz equation

$$\nabla^2 \varphi_i + \lambda_i^2 \varphi_i = 0. \tag{2.7}$$

Here λ_i^2 are the eigenvalues and $\varphi_i(x,y)$ are the corresponding two-dimensional eigenfunctions. Completeness can be shown. From Eq. (2.7) construct $(\varphi_j \nabla^2 \varphi_i - \varphi_i \nabla^2 \varphi_j)$ and integrate over the cross section Ω

$$\iint_\Omega (\varphi_j \nabla^2 \varphi_i - \varphi_i \nabla^2 \varphi_j) d\Omega = (\lambda_j^2 - \lambda_i^2) \iint_\Omega \varphi_i \varphi_j d\Omega. \tag{2.8}$$

Using Green's second identity the left hand side of Eq. (2.8) becomes

$$\oint_S (\varphi_j \nabla \varphi_i - \varphi_i \nabla \varphi_j) \cdot dS = 0. \tag{2.9}$$

The boundary integral is zero because φ_i, φ_j are zero on S. Thus from Eq. (2.8) we conclude eigenfunctions corresponding to different eigenvalues are orthogonal.

Eqs. (2.5), (2.6) give

$$1 = \sum_i A_i \varphi_i. \tag{2.10}$$

Multiply Eq. (2.10) by φ_j and integrate over the area yield the coefficients

$$A_i = \frac{\iint_\Omega \varphi_i d\Omega}{\iint_\Omega \varphi_i^2 d\Omega}. \tag{2.11}$$

The normalized total amount of heat per depth in the cylinder is

$$H = \iint_\Omega T d\Omega = \sum_i A_i \iint_\Omega \varphi_i d\Omega e^{-\lambda_i^2 t}. \tag{2.12}$$

As an example, consider the transient heat loss from a rectangular cylinder (Fig. 1(a)). Let

$$\alpha_m = (m - 1/2)\pi, \quad \beta_n = (n - 1/2)\pi / b. \tag{2.13}$$

The eigenvalues and eigenfunctions are

$$\lambda_i^2 = \alpha_m^2 + \beta_n^2, \quad \varphi_i = \cos(\alpha_m x)\cos(\beta_n y). \tag{2.14}$$

From Eq. (2.11)

$$A_i = \frac{4(-1)^{m+n}}{b \alpha_m \beta_n}. \tag{2.15}$$

On the other hand, the rectangular geometry is also amenable to separation of variables. The solution T can be written as a product of two separated solutions [1]

$$T = X(x,t) Y(y,t), \tag{2.16}$$

where

$$X = \sum_m \frac{2(-1)^{m+1}}{\alpha_m} \cos(\alpha_m x) e^{-\alpha_m^2 t}, \quad Y = \sum_n \frac{2(-1)^{n+1}}{b\beta_n} \cos(\beta_n x) e^{-\beta_n^2 t}. \tag{2.17}$$

The solution Eq. (2.16) is identical to Eq. (2.6), except in Eq. (2.16) the sum is over all positive m and n, and in Eq. (2.6) the sum is over all distinct λ_i, which excludes identical eigenfunctions.

In what follows we study some specific triangular cylinders whose cross sectional geometries are not separable. However these shapes have exact eigenvalues and eigenfunctions for the Helmholtz equation in membrane vibrations [2]. Thus exact analytic solutions for the heat diffusion problem can be obtained.

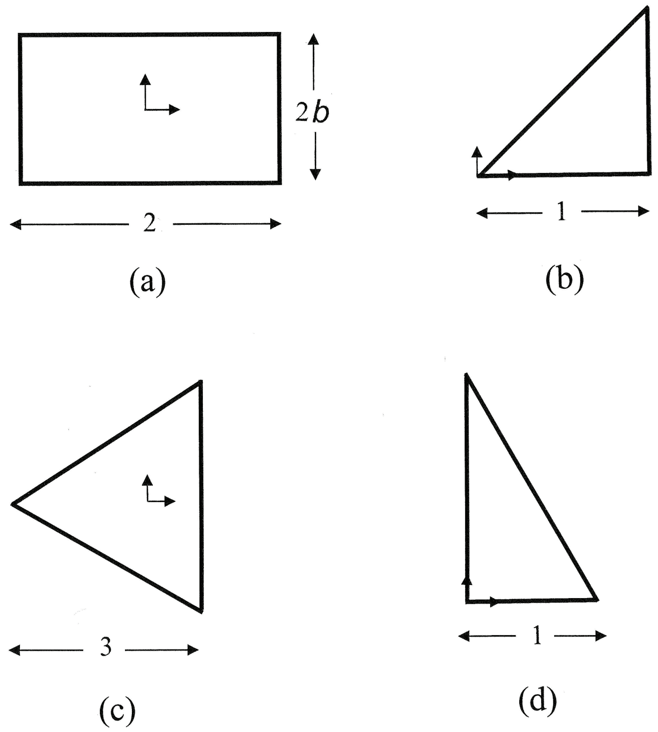

Figure 1: Cross sections of solid cylinders showing normalized length scale and coordinate axes. (a) rectangular, (b) isosceles right triangular, (c) equilateral triangular, (d) $30^\circ-60^\circ-90^\circ$ triangular.

3 The isosceles right triangular cylinder

Fig. 1(b) shows the cross section of the isosceles right triangular cylinder which has a short side as the length scale. The eigenvalues and eigenfunctions are (e.g., [2])

$$\lambda_i^2=\pi^2(m^2+n^2), \quad \varphi_i=\sin(n\pi x)\sin(m\pi y)-\sin(m\pi x)\sin(n\pi y), \tag{3.1}$$

where m, n are non-equal, non-zero positive integers. Since λ_i^2 are distinct, $m>n>0$. Eq. (2.11) gives

$$A_i=\frac{-16}{(m^2-n^2)\pi^2}\begin{cases} m/n, & m \text{ even}, n \text{ odd}, \\ n/m, & m \text{ odd}, n \text{ even}, \\ 0, & m+n=\text{even}. \end{cases} \tag{3.2}$$

The solution is thus

$$T=\sum_{i=1,\ (m>n>0)}^{N} A_i\varphi_i(x,y)e^{-\lambda_i^2 t}. \tag{3.3}$$

Table 1: The convergence of H at various times t. Empty entries signify the value has converged.

t/N	2	4	6	8	10	12	14	16
0.002	0.2646	0.3216	0.3371	0.3416	0.3429	0.3432	0.3433	0.3433
0.005	0.2282	0.2614	0.2661	0.2666	0.2667	0.2667		
0.01	0.1783	0.1923	0.1929	0.1929				

Table 2: The total heat H and the maximum temperature for the isosceles right triangular cylinder.

t	0	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
H	0.500	0.385	0.343	0.267	0.193	0.112	0.025	0.002	0.000
T_m	1.000	1.000	1.000	0.990	0.888	0.595	0.141	0.120	0.000

Here the infinite sum is truncated to N eigenvalues. Numerical convergence is tested by the total amount of heat retained Eq. (2.12) in Table 1. We see that about 16 terms give a four-digit accuracy.

Table 2 shows the total heat H and the maximum temperature (occuring on the symmetry line) for various times. It is seen that at about $t=0.2$ the cylinder is effectively cooled.

Fig. 2 shows typical temperature distribution for different times. Note that for very small times the temperature has a plateau, such that the interior is not affected by by the loss of heat from the boundary. This is also reflected in Table 2, where the maximum temperature remains at 1.000 for a finite stretch of time.

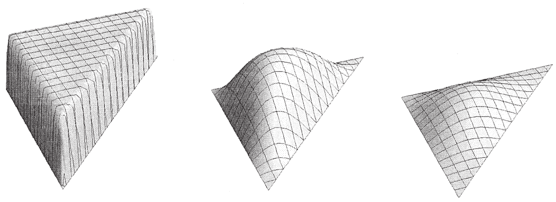

Figure 2: Temperature distribution for the isosceles right triangular cylinder. From left, $t=0.001$, $t=0.01$, $t=0.03$.

4 The equilateral triangular cylinder

Fig. 1(c) shows the equilateral triangle where the length scale is one third of tip to base height. The cylinder is enclosed by the boundaries

$$x=1, \quad y=\pm(x+2)/\sqrt{3}, \tag{4.1}$$

Schelkunoff [3] gave the eigenfunctions for the Helmholtz equation inside the equilateral triangular region

$$\varphi_i=\cos\left[\frac{(m+2n)\pi y}{3\sqrt{3}}\right]\sin\left[\frac{m\pi(2+x)}{3}\right]-\cos\left[\frac{(m-n)\pi y}{3\sqrt{3}}\right]\sin\left[\frac{(m+n)\pi(2+x)}{3}\right]$$

$$+\cos\Big[\frac{(2m+n)\pi y}{3\sqrt{3}}\Big]\sin\Big[\frac{n\pi(2+x)}{3}\Big] \tag{4.2}$$

with the eigenvalues

$$\lambda_i^2=\frac{4\pi^2}{27}\big(m^2+mn+n^2\big). \tag{4.3}$$

Here m and n are integers which are non-zero, and do not add up to zero. Schelkunoff's solution is symmetric about the x-axis. The solution for heat conduction is more restricted, that it must also have three-fold rotational symmetry. A rotation of 120° about the origin is given by the transform

$$\bar{x}=-x/2-\sqrt{3}y/2,\quad \bar{y}=\sqrt{3}x/2-y/2. \tag{4.4}$$

We require

$$\varphi_i(x,y)=\varphi_i(\bar{x},\bar{y}). \tag{4.5}$$

This yields the restriction

$$m\cos\Big(\frac{2m\pi}{3}\Big)-(m+n)\cos\Big(\frac{2(m+n)\pi}{3}\Big)+n\cos\Big(\frac{2n\pi}{3}\Big)=0. \tag{4.6}$$

We also found, for $m\neq n$,

$$\iint_\Omega \varphi_i d\Omega=0. \tag{4.7}$$

Thus the only admissible integers are $i=m=n=1,2,3,\cdots$, from which we find

$$\iint_\Omega \varphi_i d\Omega=\frac{9\sqrt{3}}{i\pi},\quad \iint_\Omega \varphi_i^2 d\Omega=\frac{9\sqrt{3}}{2}. \tag{4.8}$$

The eigenfunctions and eigenvalues are simplified to

$$\varphi_m=2\cos\Big(\frac{m\pi y}{\sqrt{3}}\Big)\sin\Big(\frac{m\pi(2+x)}{3}\Big)-\sin\Big(\frac{2m\pi(2+x)}{3}\Big), \tag{4.9a}$$

$$\lambda_m^2=\frac{4m^2\pi^2}{9}. \tag{4.9b}$$

The transient solution is thus

$$T=\sum_{m=1}^{N}\frac{2}{m\pi}\varphi_m(x,y)e^{-\lambda_m^2 t}. \tag{4.10}$$

The convergence rate and temperature distribution are similar to those of the isosceles right triangular cylinder. Table 3 shows the results.

Table 3: The total heat H and the maximum temperature for the equilateral triangular cylinder.

t	0	0.01	0.02	0.05	0.1	0.2	0.5	1	2
H	5.196	4.093	3.676	2.921	2.181	1.338	0.353	0.039	0.000
T_m	1.000	1.000	1.000	0.995	0.924	0.663	0.184	0.021	0.000

5 The $30^\circ-60^\circ-90^\circ$ triangular cylinder

Fig. 1(d) shows the cross section. The length scale used is the short side. The solution to the corresponding Helmholtz equation was found by Seth [4], here modified for our geometry. The eigenfunctions are

$$\begin{aligned}\varphi_i=&\cos\Big[\frac{(3+2m+4n)\pi}{6}(3+2x)\Big]\cos\Big[\frac{(1+2m)\pi}{2}\Big(1+\frac{2y}{\sqrt{3}}\Big)\Big]\\&-\cos\Big[\frac{(3+2n+4m)\pi}{6}(3+2x)\Big]\cos\Big[\frac{(1+2n)\pi}{2}\Big(1+\frac{2y}{\sqrt{3}}\Big)\Big]\\&+\sin\Big[\frac{(m-n)\pi}{3}(3+2x)\Big]\sin\Big[(1+m+n)\pi\Big(1+\frac{2y}{\sqrt{3}}\Big)\Big].\end{aligned}\tag{5.1}$$

The eigenvalues are

$$\lambda_i^2=\frac{4\pi^2}{9}[4(m^2+mn+n^2)+6(m+n)+3].\tag{5.2}$$

Here $m>0$, $m\neq n$ and $m\geq|n|$. In terms of increasing eigenvalues (increasing i) the (m,n) pairs are (1,−1), (1,0), (2,−2), (2,−1), (2,0), (3,−2), (3,−3), (3,−1), (2,1), (3,0), (4,−3), (4,−2), (4,−4), etc.

We find $\iint_\Omega \varphi_i d\Omega$ is exact, but too complicated to be presented here. However,

$$\iint_\Omega \varphi_i^2 d\Omega=\frac{3\sqrt{3}}{8},\quad A_i=\frac{8}{3\sqrt{3}}\iint_\Omega \varphi_i d\Omega.\tag{5.3}$$

The transient solution is

$$T=\sum_{i=1}^{N}A_i\varphi_i(x,y)e^{-\lambda_i^2 t}.\tag{5.4}$$

Table 4 shows the results.

Table 4: The total heat H and the maximum temperature T_m for the $30^\circ-60^\circ-90^\circ$ triangular cylinder.

t	0	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
H	0.866	0.706	0.645	0.532	0.419	0.285	0.106	0.023	0.001
T_m	1.000	1.000	1.000	0.999	0.972	0.810	0.352	0.077	0.004

6 Discussions

The boundaries of triangular cross sections do not conform to spatially separable coordinate systems, invalidating traditional analytic methods. Numerical solutions such as finite differences and finite elements can be used, but with much more effort, especially for a transient problem.

Using the analogy between transient heat conduction and membrane vibration, we are able to deduce exact analytic solutions for the cooling of three triangular cylinders. We are fortunate that these cross sections have exact eigenfunctions in membrane vibrations. For other shapes not described by separable coordinates exact analytic solutions have not been found.

In order to compare the three triangular cylinder solutions, a common length scale should be used. Let the new length scale be the square root of the cross sectional area, i.e.,

$$\bar{L}=\sqrt{area}, \quad \bar{t}=\frac{t'}{k\bar{L}^2}=t\frac{L^2}{\bar{L}^2}, \quad \bar{H}=H\frac{L^2}{\bar{L}^2}. \tag{6.1}$$

Fig. 3 shows a comparison of the maximum temperature and the total heat in the overbar length scale. Aside from the three triangular cylinders, we also included the known solutions of the circular cylinder (length scale is the radius)

$$T=\sum_i \frac{2}{\delta_i J_1(\delta_i)} J_0(\delta_i r) e^{-\delta_i^2 t}. \tag{6.2}$$

Here J_0, J_1 are Bessel functions of the first kind, and δ_i is the i^{th} zero of J_0. The total heat is

$$H=\sum_i \frac{4\pi}{\delta_i^2} e^{-\delta_i^2 t}. \tag{6.3}$$

The separable solution for the square cylinder, Eq. (2.16), is also plotted in Fig. 3. It is seen that the lower the perimeter to area ratio, the slower the the diffusion of heat (corresponding to the lowest fundamental frequency in vibrating membrane analogy). Thus the circular cylinder has the slowest rate of heat loss while the $30^\circ-60^\circ-90^\circ$ triangular cylinder has the fastest rate of heat loss. Although this sounds obvious, the actual heat loss or temperature distribution still depend on the analytic solutions presented here.

For a triangular cylinder of finite height, we can use the above solutions for the infinite cylinder and multiply with the parallel plate diffusion solution. Suppose the height is $2cL$. The parallel plate solution in the normal z direction is

$$T_z=\sum_l \frac{2(-1)^{l+1}}{c\gamma_l} \cos(\gamma_l z) e^{-\gamma_l^2 t}. \tag{6.4}$$

Here $l=1,2,3,\cdots$, and $\gamma_l=(l-1/2)\pi/c$. For unsteady boundary conditions, Duhamel's theorem can be applied.

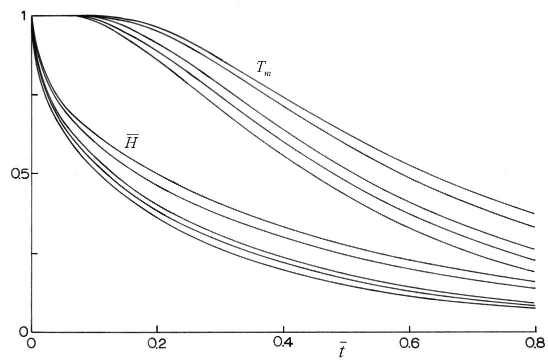

Figure 3: The maximum temperature T_m and the total heat $\bar{H}$ as a function of time $\bar{t}$ (see Eq. (6.1)). For each bunch of curves from top: cylinders with cross section of circle, square, equilateral triangle, isosceles right triangle and $30^\circ - 60^\circ - 90^\circ$ triangle.

7 Conclusions

The present paper is a fundamental contribution to the basic problem of transient heat diffusion. The solution of the diffusion problem is found to be related to the vibration of a membrane of a similar shape.

References

[1] H. S. CARSLAW AND J. C. JAEGER, Conduction of Heat in Solids, 2nd Ed., Clarendon, Oxford, 1993.
[2] C. Y. WANG AND C. M. WANG, Structural Vibration-Exact Solutions for Strings, Membranes Beams and Plates, CRC Press, Boca Raton, Florida, 2014.
[3] S. A. SCHELKUNOFF, Electromagnetic Waves, Van Nostrand, New York, 1943.
[4] B. R. SETH, *Transverse vibrations of rectilinear plates*, Proc. Indian Acad. Sci. A, 25 (1947), pp. 25–29.